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A KEY

TO

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BY

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AUTHOR OF 'PRINCIPLES OF GEOMETRY AND MENSURATION ETC.,'
'FIRST PRINCIPLES OF ARITHMETIC,' 'A SYSTEM OF MENTAL ARITHMETIC,' ETC.

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KEY

TO

TATE'S ALGEBRA MADE EASY.

1. DEFINITIONS AND NOTATION.

1. $a+2b-c=2+2\times 4-6=2+8-6=4.$

2. $a\,b+2c=2\times 4+2\times 6=8+12=20.$

3. $\frac{b}{2}+\frac{c}{3}=\frac{4}{2}+\frac{6}{3}=2+2=4.$

4. $(a+b)\,c=(2+4)\times 6=6\times 6=36.$

5. $a^2+2b-c=2\times 2+2\times 4-6=4+8-6=6.$

6. $(a-b)5+n=(3-2)5+2=5+2=7.$

7. $a^3-b^2+b+a=3\times 3\times 3-2\times 2+2+3=27-4+5=28.$

8. $(b+c)^2+a=(2+5)^2+3=7\times 7+3=52.$

9. $\frac{a+c}{2}+b=\frac{3+5}{2}+2=4+2=6.$

4. PROBLEMS IN SIMPLE EQUATIONS.

6. Let x=no. pounds left to the second; then
$3x$= „ „ first;

$\therefore\; x+3x=£20$

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$\therefore\; 4x=£20$

$\therefore\; x=\frac{1}{4}$ of $£20=£5$;

and $3x=3$ times $£5=£15$.

8. Let x=no. pence; then $3x=15d.$; and $x=\frac{1}{3}$ of $15d.$ $=5d.$

9. Let x=the number; then $9x=6$; and $x=\frac{1}{9}$ of $6=\frac{6}{9}=\frac{2}{3}$.

18. Let the 1st person's share $=x$; then
2nd „ „ $=2x$
3rd „ „ $=2$ times $2x=4x$; but the sum of all these shares is $35d.$

$$\therefore\ x+2x+4x=35d.$$
$$\therefore\ 7x=35d.$$
$\therefore\ x=\frac{1}{7}$ of $35d.=5d.$, 1st person's share; the 2nd person's share $=5d.\times2=10d.$; and the 3rd person's share $=10d.\times2=20d.$

19. Let the 1st part $=x$; then
2nd „ $=5x$
3rd „ $=5x-x=4x$; but the sum of all these parts is 20,

$$\therefore\ x+5x+4x=20$$
$$\therefore\ 10x=20$$
$\therefore\ x=\frac{1}{10}$ of $20=2$, the 1st part; the 2nd part $=5x=5\times2=10$; and the 3rd part $=4x=4\times2=8$.

21. Let x=no. in each lot; then
$12x$=value of the 1st lot, in shillings
$13x$= „ „ 2nd „ „
but the whole cost £10, or 200 shillings,

$$\therefore\ 12x+13x=200.$$
$$\therefore\ 25x=200;\ \therefore\ x=\tfrac{200}{25}=8.$$

22. Let x=no. lbs. of sugar; then
$8x$=the value of the sugar, in pence; but the value of the tea $=6s.\times4=24s.=288$ pence,

$$\therefore\ 8x=288$$
$\therefore\ x=\frac{288}{8}=36$, the no. lbs.

23. Let x=the number; then $5x=35+5=40$; $\therefore x=\frac{40}{5}=8$.

24. Let x=no. of Thomas's; then
$5x$= „ John's
$5x-x$=the difference of their numbers; but this difference is 12,

$$\therefore 5x-x=12$$
$$\therefore 4x=12$$
$$\therefore x=\frac{12}{4}=3, \text{ no. Thomas's}$$
$$\text{and } 5x=5\times 3=15, \text{ no. John's.}$$

25. Let x=the number of each; then
x=the value of the sheep, in pounds
$3x$= „ cows „
$5x$= „ horses „
but the value of the whole is £18,

$$\therefore x+3x+5x=18$$
$$\therefore 9x=18$$
$$\therefore x=\frac{18}{9}=2, \text{ no. of each.}$$

Examples in Numerical Equations.

1. $7x=21$; $\therefore x=\frac{21}{7}=3$.
2. $5x+x=24$; $\therefore 6x=24$; $\therefore x=\frac{24}{6}=4$.
3. $x+2x+3x=12$: $\therefore 6x=12$; $\therefore x=\frac{12}{6}=2$.
4. $7x=40+16$; $\therefore 7x=56$; $\therefore x=\frac{56}{7}=8$.
5. $8x-3x=15$; $\therefore 5x=15$; $\therefore x=\frac{15}{5}=3$.
6. $7x+2x-4x=25$; $\therefore 5x=25$; $\therefore x=\frac{25}{5}=5$.
7. $9x-2x=24-3$; $\therefore 7x=21$; $\therefore x=\frac{21}{7}=3$.

5. PROBLEMS IN SIMPLE EQUATIONS.

4. Let x = the less; then $x+2$=the greater; but the two parts together make 12,

$$\therefore x+x+2=12$$
$$\therefore 2x=10;$$
$$\therefore x=\frac{10}{2}=5, \text{ the less; and the greater}=x+2=5+2=7.$$

6. Let $x=$ no. for the unsuccessful candidate ; then
$x+78=$ „ successful „
but the two taken together make 864 ;

$$\therefore x+x+78=864$$
$$\therefore \quad 2x=786 ; \therefore x=\tfrac{786}{2}=393 ;$$
$$\text{and } x+78=393+78=471.$$

7. Let $x=$ the distance in miles; and
$x+8=$ the distance 8 miles farther; but the latter distance is 3 times the former,

$$\therefore 3x=x+8$$
$$\therefore 2x=8$$
$$\therefore \quad x=\tfrac{8}{2}=4, \text{ distance in miles.}$$

8. Let $x=$the price of the sugar per lb., in pence; then
$3x=$value of the sugar, in pence
$3x+4=$value of the sugar and rice, in pence ; but, from the question, this amounted to 25 pence,

$$\therefore 3x+4=25$$
$$\therefore 3x=25-4=21$$
$$\therefore \quad x=\tfrac{21}{3}=7d., \text{ price per lb.}$$

9. Let $x=$the price of each, in shillings ; then
$7x=$ „ 7 sheep „
$5x=$ „ 5 sheep „

$$\therefore 7x-5x=24s.$$
$$\therefore 2x=24s.$$
$$\therefore \quad x=\tfrac{24}{2}s.=12s., \text{ price of each.}$$

12. Let $x=$A's share, in pounds ; then
$x+8=$B's „ „
$x+8+4=$C's „ „

$$\therefore x+\overline{x+8}+\overline{x+8+4}=35$$
$$\therefore 3x+20=35$$
$$\therefore 3x=35-20=15$$
$$\therefore \quad x=\tfrac{15}{3}=\pounds 5, \text{ A's ;}$$

then B's$=x+8=5+8=\pounds 13$; and C's$=x+8+4=\pounds 17$

13. Let $x=$ James' share, in pence; then
$x+4=$ Tom's „ „
$x+4+2=$ Matthew's „ .

$$\therefore x+\overline{x+4}+\overline{x+4+2}=19$$
$$\therefore 3x+10=19$$
$$\therefore 3x=19-10=9$$
$$\therefore x=\tfrac{9}{3}=3d.,\ \text{James' share};$$

then Tom's $=3d.+4d.=7d.$; and Matthew's $=7d.+2d.=9d.$

16. Let $x=$ no. sold in pence; then
$3x=$ selling price at $3d.$ each; but he gained $10d.$ by this,

∴ *cost price in pence* $=3x-10.$

$2x=$ selling price at $2d.$ each; but he gained $1d.$ by this,

$$\therefore 3x-10=2x-1$$
$$\therefore 3x=2x+9$$
$$\therefore x=9,\ \text{no. of oranges.}$$

17. Let $x=$ no. oranges; then
$3x=$ the value of the oranges at $3d.$ each; but this is the amount of my money, that is, *my money* $=3x$
$2x=$ the value of the oranges at $2d.$ each; but my money is $7d.$ more than this,

∴ *My money* $=2x+7$

$$\therefore 3x=2x+7$$
$$\therefore x=7,\ \text{no. oranges.}$$

19. Let $x=$ no. purchased of each; then
$9x=$ cost price, in shillings, of the 1st lot
$13x=$ „ „ „ 2nd „
∴ cost price of the whole, in shillings $=9x+13x=22x$; but
selling „ „ „ „ $=2x\times 12=24x.$
selling price $-$ cost price $=$ gain,

$$\therefore 24x-22x=16$$
$$\therefore 2x=16,\ \text{total number.}$$

6. EXAMPLES IN NUMERICAL EQUATIONS.

2. $2x+4=12$; $\therefore 2x=8$; $\therefore x=\frac{8}{2}=4$.
3. $3x+2=18+x$; $\therefore 2x+2=18$; $\therefore 2x=16$; $\therefore x=8$.
4. $7x-4=31$; $\therefore 7x=35$; $\therefore x=\frac{35}{7}=5$.
5. $4x-3=x+9$; $\therefore 4x=x+12$; $\therefore 3x=12$; $\therefore x=4$.
6. $8x-4=3x+26$; $\therefore 8x=3x+30$; $\therefore 5x=30$; $\therefore x=6$.
7. $5x-8=10-x$; $\therefore 6x-8=10$; $\therefore 6x=18$; $\therefore x=3$.
8. $2x-x=14-6x$; $\therefore x=14-6x$; $\therefore 7x=14$; $\therefore x=2$.

10. PROBLEMS IN SIMPLE EQUATIONS.

3. Let x=no. Thomas's marbles at first; then
$3x$=no. John's „

but 6 being given to each, $x+6$ will then be Thomas's number, and $3x+6$ John's number; but the latter is twice the former,

$\therefore 3x+6=(x+6)\,2$
$\therefore 3x+6=2x+12$
$\therefore x=6$, no. Thomas's;
and no. John's$=6\times 3=18$.

5. Let x=no. of each at first; then
$x+8$=no. of A's after receiving 8
$x+1$= „ B's „ 1;
but the former is twice the latter,
$\therefore (x+1)\,2=x+8$
$\therefore 2x+2=x+8$
$\therefore x=6$, no. required.

7. Let x=the less; then
$x+2$=the greater,
$\therefore (x+2)\,3-2x=9$; by multiplying, we get
$\therefore 3x+6-2x=9$
$\therefore x+6=9$
$\therefore x=9-6=3$, the less;
and the greater$=3+2=5$.

9. Let x=the no. lbs. of tea at 6*s.*; then
$6x$=the value of this tea, in shillings.

Value of the second kind of tea, in shillings$=3\times 10=30$;

$\therefore$ *total value of the mixture*$=6x+30$.

Again, $x+10=$no. lbs. in the mixture; but this mixture is to be sold at 4*s*. per lb.,

$\therefore$ *total value of the mixture*$=(x+10)\ 4$;

$\therefore\ 6x+30=(x+10)\ 4$; performing the multiplication, we get
$\therefore\ 6x+30=4x+40$; by transposition, we get
$\therefore\ 6x-4x=40-30$
$\therefore\ 2x=10$; and $x=5$, the no. lbs. required.

11. Let $x=$the no. gals. of spirits; then
$x+6=$ „ „ in the mixture
$(x+6)\ 8=$value of the mixture, in shillings
$24x=$ „ „ „
$\therefore\ 24x=(x+6)\ 8$
$\therefore\ 24x=8x+48$
$\therefore\ 16x=48$; $\therefore\ x=\frac{48}{16}=3$, no. gals.

14. Let $x=$the number; then
$(x+6)\ 7=63$; dividing by 7
$x+6=\ 9$; $\therefore\ x=9-6=3$.

15. Let $x=$no. pence given to each; then
$(x+3)\ 5=25$; dividing by 5
$x+3=\ 5$; $\therefore\ x=5-3=2d$.

16. Let $x=$the breadth in feet; then
$6\ (x+2)=$area rectangle, in sq. ft.; but this area is 30 sq. ft.,
$\therefore\ 6\ (x+2)=30$; dividing by 6
$x+2=\ 5$; $\therefore\ x=5-2=3$ ft.

17. Let $x=$the less; then
$x+3=$the greater,
$\therefore\ 3x=(x+3)\ 2$; performing the multiplication, we get
$3x=2x+6$
$\therefore\ x=6$, the less; and the greater$=6+3=9$.

19. Let $x=$ no. lbs. of sugar; then
$x+3=$ „ „ increased by 3 lbs.
$(x+3)6=$ value of this sugar, in pence;
but by the question, the value of this sugar is $6s.=72d.$
$\therefore (x+3)\,6=72$; dividing by 6
$x+3=12$
$\therefore x=12-3=9$, no. lbs.

21. Let $x=$ no. stones in the 1st case; then
$x+80=$ „ „ 2nd „
No. sq. ft. in each stone in the 1st case $=4\times3=12$
„ „ „ 2nd „ $=4\times2=8$
No sq. ft. in the yard $=12x$
„ „ „ $=8\,(x+80)$
$\therefore 12x=8\,(x+80)$
$\therefore 12x=8x+640$
$\therefore 4x=640$; $\therefore x=\frac{640}{4}=160$, no. stones.

22. Let $x=$ A's share, in pounds; then
$x+2=$ B's „ „
$4\,(2x+2)=$ C's „ „
but the sum of these shares is £60;
$\therefore x+\overline{x+2}+4\,(2x+2)=60$; performing the multiplication, &c.
$\therefore 2x+2+8x+8=60$; collecting the x's, &c.
$\therefore 10x+10=60$
$\therefore 10x=50$
$\therefore x=$ £5, A's share;
B's share $=$ £5 $+$ £2 $=$ £7; and C's share $=4\,(5+7)=$ £48.

13. PROBLEMS PRODUCING SIMPLE EQUATIONS.

2. Let $x=$ no. at first; then
$x-8=$ no. remaining; but 3 times the latter are equal to the former,
$\therefore 3\,(x-8)=x$; performing the multiplication,
$\therefore 3x-24=x$; by transposition, we get
$3x-x=24$;
$\therefore 2x=24$; $\therefore x=12$, no. at first.

4. Let x=the less; then
$16-x$=the greater; then, by the question,
$3x=16-x$
$\therefore\ 4x=16$; $\therefore\ x=4$, the less;
and the greater$=16-4=12$.

5. Let x=the son's age; then
$50-x$=the father's age; but 8 times the former are equal to 2 times the latter,
$\therefore\ 8x=2\,(50-x)$; performing the multiplication,
$\therefore\ 8x=100-2x$; transposing $2x$
$\therefore\ 8x+2x=100$
$\therefore\ 10x=100$; $\therefore\ x=10$, son's age; and the father's age $=50-10=40$.

7. Let x=the no. in the 1st lot; then
$11-x=$ „ „ 2nd „
$9x$=the cost of the 1st lot, in shillings
$12\,(11-x)=$ „ „ 2nd „ „
but the cost of the two lots is £5 8*s.*=108 shillings,
$\therefore\ 9x+12\,(11-x)=108$
$\therefore\ 9x+132-12x=108$
$\therefore\ 132-3x=108$

Changing the signs of all the quantities to have the sign of the x positive or +. See Art. **8**; also Prob. 6, Art. **13.**
$3x-132=-108$
$3x=-108+132=24$
$\therefore\ x=\frac{24}{3}=8$, the no. required.

8. Let x=no. lbs of the 1st sort; then
$8-x=$ „ „ 2nd „
$7x$=cost of the 1st sort, in shillings
$4\,(8-x)=$ „ „ 2nd „ „
but the cost of the whole is 41 shillings,
$\therefore\ 7x+4(8+x)=41$
$\therefore\ 7x+32-4x=41$
$\therefore\ 3x=41-32=9$
$\therefore\ x=\frac{9}{3}=3$, no. lbs. 1st sort;
and no. lbs. of the 2nd sort$=8-3=5$.

9. Let $x=$ no. lbs. of the 1st sort; then
$12-x=$ „ 2nd „
$5x=$ cost of the 1st sort, in shillings
$3(12-x)=$ „ 2nd „ „
but the cost of the whole is £2 12*s.* $=52$ shillings,
$\therefore\ 5x+3(12-x)=52$
$\therefore\ 5x+36-3x=52$
$\therefore\ 2x=52-36=16$
$\therefore\ x=\frac{16}{2}=8$, no. lbs. 1st sort;
and, no. lbs. of the 2nd sort $=12-8=4$.

10. Let $x=$ no. days A is at work; then
$10-x=$ „ B „
$2x=$ A's wages, in shillings
$3(10-x)=$ B's „ „
but they earn the same wages,
$\therefore\ 2x=3(10-x)$
$\therefore\ 2x=30-3x$
$\therefore\ 5x=30$; $\therefore\ x=6$, no. days for A;
and no. days for B $=10-6=4$.

11. Let $x=$ price of each, in shillings
$x-2=$ „ „ at 2*s.* a head less
$8(x-2)=$ cost of the sheep at this price per head;
but the cost in this case is £4 $=80$ shillings.
$\therefore\ 8(x-2)=80$
$\therefore\ 8x-16=80$
$\therefore\ 8x=80+16=96$
$\therefore\ x=\frac{96}{8}=12$*s.*, price of each.

EXAMPLES IN NUMERICAL EQUATIONS.

3. $3(x-4)+2x=13$
$\therefore\ 3x-12+2x=13$
$\therefore\ 5x=13+12=25$
$\therefore\ x=\frac{25}{5}=5.$

4. $$4(2x+2)+3(x-1)=27$$

Performing the multiplication,

$$8x+8+3x-3=27$$

$$\therefore\ 11x+5=27$$

$$\therefore\ 11x=27-5=22$$

$$\therefore\ x=\tfrac{22}{11}=2.$$

5. $$5(x-3)+2=2(x+1)$$

$\therefore\ 5x-15+2=2x+2$; transposing, &c.

$$5x-2x=15$$

$$\therefore\ 3x=15;\ \therefore\ x=\tfrac{15}{3}=5.$$

6. $$2(4x-3)-3x=9$$

$$\therefore\ 8x-6-3x=9$$

$$\therefore\ 5x=9+6=15$$

$$\therefore\ x=\tfrac{15}{5}=3.$$

18. PROBLEMS PRODUCING SIMPLE EQUATIONS.

2. Let x = the one part, in inches; then

$\frac{x}{4}$ = the other part

but the sum of these parts is equal to 20 inches,

$$\therefore\ x+\frac{x}{4}=20$$

Multiplying both sides of the equation by 4, to get rid of the fraction, we find

$$4x+x=80$$

$\therefore\ 5x=80$; $\therefore\ x=16$ in. the one part;

and the other part = 20 in. − 16 in. = 4 in.

3. Let x = the value of my purse, in shillings; then

$\frac{x}{3}$ = " money "

but the value of these taken together is equal to 16 shillings;

$$\therefore\ x+\frac{x}{3}=16$$

To get rid of the fraction multiply both sides by 3; then

$$3x+x=48$$

$$\therefore\ 4x=48;\ \therefore\ x=12s.,\ \text{value of the purse.}$$

5. Let x=the value of the horse, in pounds; then

$\frac{x}{3}=$,, gig ,,

$$\therefore\ x+\frac{x}{3}=20$$

$$\therefore\ 3x+x=60$$

$$\therefore\ 4x=60;\ \therefore\ x=£15.$$

7. Let x = B's share, in pounds; then

$\frac{x}{3}$ = A's ,, ,,

$\frac{x}{6}$ = C's ,, ,,

but the sum of these shares is £63,

$$\therefore\ x+\frac{x}{3}+\frac{x}{6}=63.$$

To get rid of the fractions we multiply each side by 6; then

$$6x+2x+x=378$$

$$\therefore\ 9x=378$$

$$\therefore\ x=\tfrac{378}{9}=£42,\ \text{B's share;}$$

A's share=£$\frac{42}{3}$=£14; and C's share=$\frac{42}{6}$=£7.

8. Let x=the number; then

$\frac{x}{4}$=the 4th part of the number

$\frac{x}{5}=$,, 5th ,, ,,

$$\therefore\ \frac{x}{4}-\frac{x}{5}=1$$

To clear this equation of fractions we multiply each side by 20; then

$$5x-4x=20$$

$$\therefore\ x=20,\ \text{the number.}$$

9. Let x=the money at first, in pence; then
$x-8=$ „ remaining „

$\therefore x-8=\frac{x}{5}$; multiplying each side by 5

$5x-40=x$; by transposition
$5x-x=40$;
$\therefore 4x=40$; $\therefore x=10d.$

10. Let x = cost price of the sheep, in shillings; then
$15-x=$ *gain, in shillings*

$\frac{x}{2}=$ *gain, in shillings*

$\therefore \frac{x}{2}=15-x$; multiplying each side by 2

$x=30-2x$; adding $2x$ to each side
$3x=30$; $\therefore x=10s.$, cost price required.

11. Let x =the number; then

$$\frac{x}{3}+4=9$$

$$\therefore \frac{x}{3}=9-4=5$$

$$\therefore x=5\times3=15, \text{ the number.}$$

12. Let x=the number; then

$\frac{x}{2}-2=\frac{x}{3}$; multiplying each side by 6

$3x-12=2x$; by transposition
$3x-2x=12$;
$\therefore x=12$, the number.

14. Let x=the number; then

$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}-x=2$; multiplying each side by 12

$6x+4x+3x-12x=24$
$\therefore x=24$, the number.

15. Let x=the length of the post, in feet; then

$\frac{x}{10}$= portion in the water ,,

$\frac{x}{4}$= ,, ,, ground ,,

13= ,, ,, above the ground.

But the sum of all these portions makes up the whole length of the post;

$\therefore\ x=\frac{x}{10}+\frac{x}{4}+13$; multiplying each side by 20

$20x=2x+5x+260$

$\therefore\ 13x=260$

$\therefore\ x=\frac{260}{13}=20$ feet, the length required.

17. Let x =his income, in pounds; then

$\frac{x}{4}$ =money spent ,,

$x-\frac{x}{4}$ =money left ,,

But, by the question, this is equal to $10+\frac{x}{8}$,

$\therefore\ x-\frac{x}{4}=10+\frac{x}{8}$; multiplying by 8

$8x-2x=80+x$

$\therefore\ 5x=80$

$\therefore\ x=\frac{80}{5}=$ £16, income required.

18. Let x=my money, in pence; then

$\frac{x}{7}$=one-seventh of my money in pence

$\therefore\ x-\frac{x}{7}=10+2=12$; multiplying by 7

$7x-x=84$

$\therefore\ 6x=84$; $\therefore\ x=\frac{84}{6}=14$*d.*

22. Let x = my age; then

$$\frac{x}{5} = \text{John's age}$$

$$\tfrac{1}{2} \text{ of } \frac{x}{5} = \text{Tom's age}$$

$$\therefore\ x+\frac{x}{5}+\tfrac{1}{2} \text{ of } \frac{x}{5} = 39$$

$$\therefore\ x+\frac{x}{5}+\frac{x}{10} = 39 \text{ ; multiplying by 10}$$

$$10x+2x+x=390$$

$$\therefore\ 13x=390$$

$$\therefore\ x=\tfrac{390}{13}=30, \text{ my age ;}$$

John's age $=\frac{30}{5}=6$; and Tom's age $=\frac{1}{2}$ of $6=3$.

Or thus: To avoid fractions, let $10x$ = my age; then John's age $=\frac{1}{5}$ of $10x=2x$; and Tom's age $=\frac{1}{2}$ of $2x=x$.

$$\therefore\ 10x+2x+x=39$$

$$\therefore\ 13x=39 \text{ ; } \therefore\ x=3$$

$$\therefore\ \text{My age}=10x=10\times 3=30.$$

23. Let x = the total number of trees; then

$\frac{x}{3}$ = the number of apple trees

$\frac{x}{4}$ = „ pear „

35 = „ plum „

But the total number of trees in the garden is equal to the sum of these different kinds of trees;

$$\therefore\ x=\frac{x}{3}+\frac{x}{4}+35 \text{ ; multiplying each by 12}$$

$$12x=4x+3x+420$$

$$\therefore\ 5x=420 \text{ ; } \therefore\ x=\tfrac{420}{5}=84, \text{ the no. trees.}$$

19. EXAMPLES IN NUMERICAL EQUATIONS.

3. $\frac{x}{3}+\frac{x+2}{4}=4$; multiplying each by 12

$4x+3(x+2)=48$

$\therefore\ 4x+3x+6\ =48$

$\therefore\ 7x=42$; $\therefore\ x=\frac{42}{7}=6$.

4. $\frac{x}{4}+\frac{x}{5}=\frac{x}{8}+13$; multiplying each by 40

$10x+8x=5x+520$

$\therefore\ 13x=520$; $\therefore\ x=\frac{520}{13}=40$.

5. $\frac{x+1}{5}=\frac{x+2}{6}+1$; multiplying each by 30

$6(x+1)=5(x+2)+30$

$\therefore\ 6x+6=5x+10+30$

$\therefore\ \quad x\ =10+30-6=34$.

6. $\frac{x}{2}+\frac{x}{4}+x=\frac{x}{3}+17$; multiplying each by 12

$6x+3x+12x=4x+204$

$\therefore\ 17x=204$; $\therefore\ x=\frac{204}{17}=12$.

7. $x-\frac{x}{2}+8=20$

$\therefore\ x-\frac{x}{2}=12$; multiplying each by 2

$2x-x=24$

$\therefore\ x=24$.

8. $2x-\frac{x}{3}-4=28-x$; by transposition

$2x-\frac{x}{3}+x=28+4$

$\therefore\ \ 3x-\frac{x}{3}=32$; multiplying each by 3

$9x-x=96$

$\therefore\ 8x=96$; $\therefore\ x=\frac{96}{8}=12$.

9. $\frac{x}{2}+\frac{x}{3}-\frac{x}{4}=\frac{x}{4}+5$; multiplying each by 12

$6x+4x-3x=3x+60$;

$\therefore\ 4x=60$; $\therefore\ x=\frac{60}{4}=15$.

23. EXAMPLES IN FRACTIONS.

10. $\frac{2x}{3}+\frac{x}{6}=\frac{4x}{6}+\frac{x}{6}=\frac{5x}{6}$.

11. $\frac{x}{2}+\frac{2x}{5}=\frac{5x}{10}+\frac{4x}{10}=\frac{9x}{10}$.

12. $\frac{x}{3}+\frac{2x}{9}=\frac{3x}{9}+\frac{2x}{9}=\frac{5x}{9}$.

14. $\frac{x}{3}+\frac{x}{4}+\frac{x}{2}=\frac{4x+3x+6x}{12}=\frac{13x}{12}$.

15. $\frac{x}{5}+\frac{x}{10}+\frac{x}{4}=\frac{4x+2x+5x}{20}=\frac{11x}{20}$.

16. $\frac{x}{3}+\frac{x}{9}-\frac{x}{6}=\frac{6x}{18}+\frac{2x}{18}-\frac{3x}{18}=\frac{5x}{18}$.

17. $x-\frac{x}{2}+\frac{x}{4}=\frac{4x}{4}-\frac{2x}{4}+\frac{x}{4}=\frac{3x}{4}$.

18. $\frac{x}{2}-\frac{x}{3}=\frac{3x}{6}-\frac{2x}{6}=\frac{x}{6}$.

19. $\frac{2x}{5}-\frac{x}{4}=\frac{8x}{20}-\frac{5x}{20}=\frac{3x}{20}$.

20. $\frac{7x}{9}-\frac{2x}{3}=\frac{7x}{9}-\frac{6x}{9}=\frac{x}{9}$.

22. $\frac{9x}{2}=\frac{8x}{2}+\frac{x}{2}=4x+\frac{x}{2}$.

23. $\frac{8x}{3}=\frac{6x}{3}+\frac{2x}{3}=2x+\frac{2x}{3}$.

24. $\frac{7x}{5}=\frac{5x}{5}+\frac{2x}{5}=x+\frac{2x}{5}$.

25. $\frac{x}{3}\div\frac{2}{3}=\frac{x}{3}\times\frac{3}{2}=\frac{x}{2}.$

26. $\frac{2x}{15}\div\frac{4a}{5}=\frac{\cancel{2}x}{\underset{3}{\cancel{15}}}\times\frac{\cancel{5}}{\underset{2}{\cancel{4}a}}=\frac{x}{6a}.$

24. PROBLEMS PRODUCING SIMPLE EQUATIONS.

3. Let x=the number; then

$\frac{x}{3}$=one-third the number

$x-\frac{x}{3}$= the remainder

Cost of the oranges in pence$=2\times\frac{x}{3}+1\frac{1}{2}\left(x-\frac{x}{3}\right)$;

but the cóst of the oranges is 15 pence,

$\therefore\ 2\times\frac{x}{3}+1\frac{1}{2}\left(x-\frac{x}{3}\right)=15$; multiplying each by 6

$$4x+9\left(x-\frac{x}{3}\right)=90$$

$$\therefore 4x+9x-3x=90$$

$$10x=90$$

$\therefore x=\frac{90}{10}=9$, the number.

4. Let x=the number; then

$\frac{5x-2}{2}-x=5$; multiplying each by 2

$$5x-2-2x=10$$

$$\therefore 3x=10+2=12$$

$$\therefore x=\frac{12}{3}=4.$$

5. Let x=the no. sheep at first; then

the no. lost$=\frac{x}{3}+4$;

$\therefore$ the no. remaining$=x-\left(\frac{x}{3}+4\right)=x-\frac{x}{3}-4.$

But as $\frac{1}{4}$ of this number was sold, there will be $\frac{3}{4}$ of it left;

$$\therefore \text{No. left}=\frac{3}{4}\left(x-\frac{x}{3}-4\right);$$

but the number left was 15;

$$\therefore \frac{3}{4}\left(x-\frac{x}{3}-4\right)=15; \text{ multiplying each by 4}$$

$$3\left(x-\frac{x}{3}-4\right)=60$$

$$\therefore 3x-x-12=60$$

$$\therefore 2x=60+12=72$$

$$\therefore x=\frac{72}{2}=36, \text{ no. at first.}$$

8. Let x=the no. of crowns; then
$13-x=$ „ sixpences.
Value of the crowns, in shillings$=5x$
„ sixpences „ $=\frac{1}{2}(13-x)$.

But the sum of these two must amount to 20 shillings;

$$\therefore 5x+\frac{1}{2}(13-x)=20; \text{ multiplying each by 2}$$

$$10x+13-x=40$$

$$\therefore 9x=40-13=27$$

$$\therefore x=\frac{27}{9}=3, \text{ no. crowns;}$$

and the no. sixpences$=13-3=10$.

9. Let x=the weight of the copper, in lbs.; then
$\frac{2x}{3}=$ „ zinc „

But the sum of these must be equal to 30 lbs.;

$$\therefore x+\frac{2x}{3}=30; \text{ multiplying each by 3}$$

$$3x+2x=90$$

$$\therefore 5x=90$$

$$x=\frac{90}{5}=18 \text{ lbs. wt. of copper;}$$

and the wt. of the zinc$=\frac{2}{3}$ of 18 lbs.$=12$ lbs.

10. Let $x=$ no. gals. in the cask; then

No. gals. first drawn off$=\frac{x}{3}$;

„ remaining $=x-\frac{x}{3}=\frac{2x}{3}$;

No. gals. next drawn off$=\frac{1}{2}$ of $\frac{2x}{3}=\frac{x}{3}$;

„ now remaining$=\frac{2x}{3}-\frac{x}{3}=\frac{x}{3}$;

but, by the question, there were 14 gals. remaining;

$$\therefore \frac{x}{3}=14;\ \therefore x=42 \text{ gals.}$$

12. Let $x=$the son's age; then
$x+32=$the father's age
$\therefore x=\frac{1}{5}(x+32)$; multiplying each by 5
$5x=x+32$
$\therefore 4x=32$; $\therefore x=8$, the son's age.

13. Let $x=$the no. persons; then

three-fourths of them$=\frac{3x}{4}$;

$\therefore$ the rest of them$=x-\frac{3x}{4}=\frac{x}{4}$.

No. shillings given to the former$=2\times\frac{3x}{4}=\frac{6x}{4}=\frac{3x}{2}$;

„ „ latter$=5\times\frac{x}{4}=\frac{5x}{4}$;

but the sum of these two must be equal to 22 shillings;

$\therefore \frac{3x}{2}+\frac{5x}{4}=22$; multiplying each by 4

$6x+5x=88$;

$\therefore x=\frac{88}{11},=8$, no. persons.

15. Let x=the distance each way, in miles; then

$\frac{x}{4}$=no. hours in going

$\frac{x}{3}$= „ „ returning.

But he takes 21 hours in going and returning;

$\therefore \frac{x}{4}+\frac{x}{3}=21$, multiplying each by 12

$$3x+4x=252$$

$$\therefore 7x=252; \therefore x=\tfrac{252}{7}=36;$$

and $2x=36\times 2=72$ miles, the total distance.

16. Let $2x$=the one part; then

$3x$=the other part

$$\therefore 2x+3x=£30;$$

$$\therefore 5x=£30; \quad \therefore x=\frac{£30}{5}=£6;$$

and the one part=$£6\times 2=£12$;

and the other part=$£6\times 3=£18$.

17. Let x=the one part, in pounds; then

$250-x$=the other part, „

Total annual interest, in pounds=$\frac{3x}{100}+\frac{4(250-x)}{100}$.

But the total annual interest is £8;

$\therefore \frac{3x}{100}+\frac{4(250-x)}{100}=8$; multiplying each by 100

$$3x+4(250-x)=800$$

$$\therefore 3x+1000-4x=800$$

$\therefore -x=800-1000=-200$; (Art. 8)

$x=£200$, the one part;

and the other part$=250-200=£50$.

18. Let x=the no. lbs. required ; then

$$\textit{Value per lb., in shillings}=\frac{80}{x}.$$

The no. lbs. lost$=\frac{1}{3}(x-5)$;

but the value of this tea$=£4-£3=20$ shillings,

$$\therefore \textit{Value per lb., in shillings}=\frac{20}{\frac{1}{3}(x-5)}=\frac{60}{x-5};$$

$$\therefore \frac{80}{x}=\frac{60}{x-5}.$$

Multiplying both sides of this equation, first by x, and then by $x-5$, we get

$$80(x-5)=60x$$

$\therefore 80x-400=60x$; by transposition, we get

$$80x-60x=400$$

$$\therefore 20x=400;\ \therefore x=20\text{lbs.}$$

20. Here when the short pointer is at 3, the long pointer is at 12, that is, at this time the former is 15 minute-spaces in advance of the latter. Let x = the no. minutes of time past 3 o'clock when the two pointers are together; then

Minute-spaces moved over by the long pointer in x min. $=x$.

" " " short " $=x-15$.

But the former is 12 times the latter;

$$\therefore 12(x-15)=x$$

$$\therefore 12x-180=x$$

$$\therefore 11x=180;\ \therefore x=\frac{180}{11}=16\frac{4}{11}\text{ min.};$$

and the time will be $16\frac{4}{11}$ min. past 3 o'clock.

25. EXAMPLES IN SUBTRACTION.

1. $5x-(2x-7)=5x-2x+7=3x+7.$
2. $6x-(3x-a)=6x-3x+a=3x+a.$
3. $2x+a-(x-2a)=2x+a-x+2a=x+3a.$
4. $x-2a\quad(x-3a)=x-2a-x+3a=a.$
5. $2x-(x-3a)=2x-x+3a=x+3a.$

6. $4x+2-(3x-4)=4x+2-3x+4=x+6.$

7. $3a-1-(a-6)=3a-1-a+6=2a+5.$

11. $\frac{x-3}{2}-\frac{x-2}{3}=\frac{3x-9}{6}-\frac{2x-4}{6}=\frac{x-5}{6}.$

12. $3x-\frac{2x-1}{2}=\frac{6x}{2}-\frac{2x-1}{2}=\frac{4x+1}{2}.$

13. $\frac{9x}{5}-\frac{x+a-2}{3}=\frac{27x}{15}-\frac{5x+5a-10}{15}=\frac{22x-5a+10}{15}.$

14. $x-\frac{2x-1}{5}=\frac{5x}{5}-\frac{2x-1}{5}=\frac{3x+1}{5}.$

15. $\frac{x+1}{2}-\frac{2x-3}{4}=\frac{2x+2}{4}-\frac{2x-3}{4}=\frac{5}{4}.$

16. $3a-\frac{a-c+2}{3}=\frac{9a}{3}-\frac{a-c+2}{3}=\frac{8a+c-2}{3}.$

26. PROBLEMS PRODUCING SIMPLE EQUATIONS.

3. Let x=the number; then we have the following equation:

$7-\frac{x-3}{5}=4$; multiplying each side by 5

$35-(x-3)=20$; completing the subtraction

$35-x+3=20$

$\therefore \quad 38-x=20$; changing all the signs

$x-38=-20$

$\therefore \; x=-20+38=18$

5. Let x=the money at first, in pounds; then

Money spent, in pounds$=10+\frac{1}{4}(x-10)$

$\therefore$ „ left, „ $=x-10-\frac{1}{4}(x-10)$;

but the money left was £30;

$\therefore \; x-10-\frac{1}{4}(x-10)=30$; adding 10 to each side

$x-\frac{1}{4}(x-10)=40$; multiplying each by 4

$4x-(x-10)=160$; completing the subtraction

$3x+10=160$;

$\therefore \; 3x=150$; $\therefore \; x=\frac{150}{3}=$£50.

7. Let x=the wages per day, in shillings.

Here we shall find two independent expressions for the number of days the labourer was at work.

$$\text{No. days at work}=\frac{10}{x-1};$$

$$\text{,,} \qquad \text{,,} \quad =\frac{20}{x+1};$$

$$\therefore \quad \frac{20}{x+1}=\frac{10}{x-1}$$

Multiplying both sides of this equation, first by $x+1$, and then by $x-1$, we get

$$20x-20=10x+10$$

$$\therefore \ 10x=30; \ \therefore \ x=3s.$$

8. Let x=the wages per day, in shillings ; then

Wages for 5 days at x shillings per day$=5x$;

„ „ 3 „ $(x-1)$ „ „ $=3(x-1)$;

but the former is 10 shillings more than the latter,

$$\therefore \ 5x-3(x-1)=10$$

$$\therefore \ 5x-3x+3=10$$

$$\therefore \ 2x=10-3=7$$

$$\therefore \ x=\tfrac{7}{2}s.=3s. \ 6d.$$

27. EXAMPLES IN NUMERICAL EQUATIONS.

2. $\frac{x-4}{3}-\frac{x-1}{4}=1$; multiplying every term by 12

$4(x-4)-3(x-1)=12$; performing the multiplication, &c.

$4x-16-3x+3=12$;

$\therefore \ x-13=12$;

$\therefore \ x=12+13=25.$

3. $x-\frac{3x-5}{4}=2$; multiplying every term by 4

$4x-(3x-5)=8$; performing the subtraction

$x+5=8$; $\therefore \ x=8-5=3.$

4. $5\,(x+2)-\frac{x-3}{2}=52$; multiplying every term by 2

$10x+20-(x-3)=104$; performing the subtraction

$9x+23=104$;

$\therefore\ 9x=104-23=81$; $\therefore\ x=9$.

5. $2x-5\,(2-x)+2=6$; performing the multiplication

$2x-(10-5x)+2=6$; performing the subtraction

$7x-8=6$

$\therefore\ 7x=6+8=14$; $\therefore\ x=2$.

6. $\frac{3x+8}{5}-\frac{x-8}{4}-2=3$

$\therefore\ \frac{3x+8}{5}-\frac{x-8}{4}=5$; multiplying every term by 20

$4\,(3x+8)-5\,(x-8)=100$; performing the multiplication, &c.

$\therefore\ 12x+32-5x+40=100$; collecting

$7x+72=100$; by transposition

$\therefore\ 7x=100-72=28$

$\therefore\ x=\frac{28}{7}=4$.

7. $\frac{5x+3}{2}-2\,(x-2)=6$; multiplying every term by 2

$5x+3-4\,(x-2)=12$;

$\therefore\ 5x+3-4x+8=12$;

$\therefore\ x+11=12$: $\therefore\ x=12-11=1$.

29. EXAMPLES IN MULTIPLICATION.

2.
$$\begin{array}{r}
2x-4\\
3x-2\\
\hline
6x^2-12x\\
-\ \ 4x+8\\
\hline
6x^2-16x+8.
\end{array}$$

3.
$$\begin{array}{r}
5a+3\\
3a-5\\
\hline
15a^2+9a\\
-25a-15\\
\hline
15a^2-16a-15.
\end{array}$$

4.
$$\begin{array}{r}
2x+3a\\
4x-3a\\
\hline
8x^2+12ax\\
-6ax-9a^2\\
\hline
8x^2+6ax-9a^2.
\end{array}$$

5.
$$\begin{array}{r}
4x+2a\\
2x-\ a\\
\hline
8x^2+4ax\\
-4ax-2a^2\\
\hline
8x^2\quad *\quad -2a^2.
\end{array}$$

6.

$$\begin{array}{llll}
x^2+xy+y^2\times(x-y) & & & \\
\quad x-y & & & \\
\hline
x^3+x^2y+xy^2 & & & \\
\quad -x^2y-xy^2-y^3 & & & \\
\hline
x^3 \quad * \quad * \quad -y^3 & & & \\
\hline
\end{array}$$

7. $$\frac{2x+1}{3}\times\frac{(1-2x)}{2}=\frac{(2x+1)\,(1-2x)}{6}=\frac{1-4x^2}{6}.$$

Here the product of the quantities in the numerator is obtained as follows:

$$\begin{array}{rl}
2x+1 & \\
1-2x & \\
\hline
2x+1 & =\text{product by } 1 \\
-4x^2-2x \quad & =\text{product by} -2x \\
\hline
-4x^2 \quad * \quad +1 & =1-4x^2. \\
\hline
\end{array}$$

8. $$\frac{3a+2}{5}\times\frac{5}{a}=\frac{(3a+2)\times 5}{5\times a}=\frac{3a+2}{a}.$$

9. $$\frac{a-b}{2}\times\frac{a+b}{2}=\frac{(a-b)\,(a+b)}{4}=\frac{a^2-b^2}{4}.$$

Here the product of the quantities in the numerator is obtained as follows:

$$\begin{array}{ll}
a-b & \\
a+b & \\
\hline
a^2-a\,b & =\text{product by } a \\
\quad a\,b-b^2 & =\text{product by } b \\
\hline
a^2 \quad * \quad -b^2. & \\
\hline
\end{array}$$

33. EXAMPLES IN DIVISION.

1.

$$\begin{array}{lll}
x-2)\ x^2-6x+8\ (x-4 & & \\
\qquad x^2-2x & =x \text{ times } (x-2) & \\
\hline
\qquad -4x+8 & & \\
\qquad -4x+8 & =-4 \text{ times } (x-2) & \\
\hline
\qquad\ * \quad * & & \\
\hline
\end{array}$$

2.
$$\begin{array}{l}
2x+1)\ 6x^2+5x+1\ (3x+1 \\
\qquad\qquad \underline{6x^2+3x} \qquad =3x \text{ times } (2x+1) \\
\qquad\qquad\qquad 2x+1 \\
\qquad\qquad\qquad \underline{2x+1}=1\text{ce } (2x+1) \\
\qquad\qquad\qquad \underline{\ast \quad \ast}
\end{array}$$

3.
$$\begin{array}{l}
5x+6)\ 5x^2+31x+30\ (x+5 \\
\qquad\qquad \underline{5x^2+\ 6x} \qquad =x \text{ times } (5x+6) \\
\qquad\qquad\qquad 25x+30 \\
\qquad\qquad\qquad \underline{25x+30}=5 \text{ times } (5x+6) \\
\qquad\qquad\qquad \underline{\ \ast \quad \ast\ }
\end{array}$$

4.
$$\begin{array}{l}
2x+3a)\ 2x^2+5ax+3a^2\ (x+a \\
\qquad\qquad \underline{2x^2+3ax} \qquad =x \text{ times } (2x+3a) \\
\qquad\qquad\qquad 2ax+3a^2 \\
\qquad\qquad\qquad \underline{2ax+3a^2}=a \text{ times } (2x+3a) \\
\qquad\qquad\qquad \underline{\ \ast \quad \ast\ }
\end{array}$$

5.
$$\begin{array}{l}
3x-2a)\ 9x^2-4a^2\ (3x+2a \\
\qquad\qquad \underline{9x^2-6ax} \qquad =3x \text{ times } (3x-2a) \\
\qquad\qquad\qquad 6ax-4a^2 \\
\qquad\qquad\qquad \underline{6ax-4a^2}=2a \text{ times } (3x-2a) \\
\qquad\qquad\qquad \underline{\ \ast \quad \ast\ }
\end{array}$$

6.
$$\begin{array}{l}
x-2)\ x^2+(a-2)\ x-2a\ (x+a \\
\qquad\qquad \underline{x^2- \qquad 2x} \qquad =x \text{ times } (x-2) \\
\qquad\qquad\qquad\qquad ax-2a \\
\qquad\qquad\qquad\qquad \underline{ax-2a}=a \text{ times } (x-2) \\
\qquad\qquad\qquad\qquad \underline{\ \ast \quad \ast\ }
\end{array}$$

7.
$$\begin{array}{l}
2x-3)\ 4x^2-12x+9\ (2x-3 \\
\qquad\qquad \underline{4x^2-\ 6x} \qquad =2x \text{ times } (2x-3) \\
\qquad\qquad\quad -\ 6x+9 \\
\qquad\qquad\quad \underline{-\ 6x+9}=-3 \text{ times } (2x-3) \\
\qquad\qquad\quad \underline{\quad \ast \quad \ast\ }
\end{array}$$

35. EXAMPLES IN POWERS AND ROOTS.

1. $b^2 \times b^2 \times b^2 = \overline{b \times b} \times \overline{b \times b} \times \overline{b \times b} = b^6.$

2. $a^3 \times a^3 \times a^3 \times a^3 =$
$\overline{a \times a \times a} \times \overline{a \times a \times a} \times \overline{a \times a \times a} \times \overline{a \times a \times a} = a^{12}.$

3.
$$\begin{array}{llll} 2 & +x & & \\ 2 & +x & & \\ \hline 4 & +2x & & =2 \text{ times } (2+x) \\ & 2x & +x^2 & =x \text{ times } (2+x) \\ \hline 4 & +4x & +x^2 & =(2+x)\times(2+x)=(2+x)^2. \end{array}$$

4.
$$\begin{array}{llll} x & +2a & & \\ x & +2a & & \\ \hline x^2 & +2ax & & =x \text{ times } (x+2a) \\ & 2ax & +4a^2 & =2a \text{ times } (x+2a) \\ \hline x^2 & +4ax & +4a^2 & =(x+2a)\times(x+2a)=(x+2a)^2. \end{array}$$

5. $(-2x)^3 = -2x \times -2x \times -2x = +4x^2 \times -2x = -8x^3.$

6. $(-3x^2)^2 = -3x^2 \times -3x^2 = 9x^4.$

7. $\left(\frac{2x}{3}\right)^2 = \frac{2x}{3} \times \frac{2x}{3} = \frac{4x^2}{9}.$

9.
$$\begin{array}{lllll} v & +\frac{2}{v} & & & \\ v & +\frac{2}{v} & & & \\ \hline v^2 & +2 & & & \\ & 2 & +\frac{4}{v^2} & & \\ \hline v^2 & +4 & +\frac{4}{v^2} & & =\left(v+\frac{2}{v}\right)\times\left(v+\frac{2}{v}\right)=\left(v+\frac{2}{v}\right)^2. \\ & v & +\frac{2}{v} & & \\ \hline v^3 & +4v & +\frac{4}{v} & & \\ & 2v & +\frac{8}{v} & +\frac{8}{v^3} & \\ \hline v^3 & +6v & +\frac{12}{v} & +\frac{8}{v^3} & =\left(v+\frac{2}{v}\right)^2\times\left(v+\frac{2}{v}\right)=\left(v+\frac{2}{v}\right)^3. \end{array}$$

Or thus. By the law of the expansion of a binomial,

$$\left(v+\frac{2}{v}\right)^3=v^3+3v^2\frac{2}{v}+3v\left(\frac{2}{v}\right)^2+\left(\frac{2}{v}\right)^3$$
$$=v^3+6v+\frac{12}{v}+\frac{8}{v^3}.$$

10.
$$\begin{array}{lll}
x+\frac{1}{x} & & \\
x+\frac{1}{x} & & \\
\hline
x^2+1 & & =\left(x+\frac{1}{x}\right)\times x\\
 & 1+\frac{1}{x^2} & =\left(x+\frac{1}{x}\right)\times\frac{1}{x}\\
\hline
x^2+2+\frac{1}{x^2} & & =\left(x+\frac{1}{x}\right)\times\left(x+\frac{1}{x}\right)=\left(x+\frac{1}{x}\right)^2.\\
\hline
\end{array}$$

Or thus. By the law of the expansion of a binomial,

$$\left(x+\frac{1}{x}\right)^2=x^2+2x\frac{1}{x}+\left(\frac{1}{x}\right)^2=x^2+2+\frac{1}{x^2}.$$

11. $(\sqrt{x})^4=\sqrt{x}\times\sqrt{x}\times\sqrt{x}\times\sqrt{x}=x\times x=x^2.$

12.
$$\begin{array}{lll}
\sqrt{x}+2 & & \\
\sqrt{x}+2 & & \\
\hline
x+2\sqrt{x} & & =(\sqrt{x}+2)\times\sqrt{x}\\
 & 2\sqrt{x}+4 & =(\sqrt{x}+2)\times 2\\
\hline
x+4\sqrt{x}+4 & & =(\sqrt{x}+2)^2.\\
\hline
\end{array}$$

Or thus. By the law of the expansion of a binomial,

$$(\sqrt{x}+2)^2=(\sqrt{x})^2+2\sqrt{x}\times 2+2^2=x+4\sqrt{x}+4.$$

14. Here, $x^8=x^4\times x^4$; $\therefore\ \sqrt{x^8}=x^4.$

15. Here, $x^{12}=x^4\times x^4\times x^4$; $\therefore\ \sqrt[3]{x^{12}}=x^4.$

16. Here, $9x^2=3x\times 3x$; $\therefore\ \sqrt{9x^2}=3x.$

17. Here, $16x^4=4x^2\times 4x^2$; $\therefore\ \sqrt{16x^4}=4x^2.$

20. $\sqrt{x^2+8x+16}=x+4.$ The square root of the first term, x^2, is x; and the square root of the last term, 16, is 4;

and moreover the middle term, $8x$, is equal to $2\times x\times 4$; therefore $x+4$ is the square root required.

Or thus.

$$\begin{array}{rl} & x^2+8x+16\ (x+4 \\ & \underline{x^2} \\ 2x+4) & *\ \ +8x+16 \\ & \qquad \underline{8x+16} \\ & \qquad\ \ *\quad * \end{array}$$

Writing down the proposed expression, the first term, x, of the root is the square root of x^2; subtracting x^2, the remainder is $8x+16$. The first term of the divisor, $2x+4$, is the *double* of x; then dividing $8x$ by $2x$ we obtain $+4$, the other term in the root; lastly, adding 4 to the $2x$, we get $2x+4$, which multiplied by 4, and subtracted, leaves no remainder. Therefore $x+4$ is the square root required.

21. $\sqrt{x^2-2x+1}=x-1$. Here x is the square root of x^2; and -1 is the square root of $+1$; and, moreover, $-2x=2\times x\times -1$.

Or thus.

$$\begin{array}{rl} & x^2-2x+1\ (x-1 \\ & \underline{x^2} \\ 2x-1) & *\ -2x+1 \\ & \quad \underline{-2x+1} \\ & \qquad *\quad * \end{array}$$

The square root of x^2 is x; and the product of the divisor, $2x-1$, by -1 gives the remainder $-2x+1$.

22. $\sqrt{x^2-4x+4}=x-2$. Here x is the square root of x^2; and -2 is the square root of $+4$; and, moreover, the middle term, $-4x=2\times x\times -2$.

Or thus.

$$\begin{array}{rl} & x^2-4x+4\ (x-2 \\ & x^2 \\ 2x-2) & *\ -4x+4 \\ & \quad \underline{-4x+4} \\ & \qquad *\quad * \end{array}$$

The square root of x^2 is x; and the product of the divisor, $2x-2$, by -2 gives the remainder $-4x+4$.

23. $\sqrt{4x^2-4x+1}=2x-1$. Here $2x$ is the square root of $4x^2$; and -1 is the square root of 1; and moreover the middle term, $-4x$, is equal to $2\times 2x\times -1$.

Or thus.

$$\begin{array}{rl}
 & 4x^2-4x+1\ (2x-1 \\
 & 4x^2 \\
4x-1) & \overline{\ast\quad -4x+1} \\
 & \quad\ \ \underline{-4x+1} \\
 & \quad\ \ \underline{\ \ \ast\qquad \ast\ \ }
\end{array}$$

The square root of $4x^2$ is $2x$; the first term of the divisor is $2x\times 2$; and the complete divisor multiplied by -1 gives the remainder $-4x+1$.

24. $\sqrt[3]{(x+y)^3}=\sqrt[3]{(x+y)\,(x+y)\,(x+y)}=x+y$.

36. EXAMPLES IN SIMPLE EQUATIONS.

3. $$x+2=\frac{x^2+16}{x+2}.$$

To clear the equation of fractions, multiply both sides by $x+2$.

$$(x+2)\,(x+2)=x^2+16.$$

Completing the multiplication,

$$x^2+4x+4=x^2+16.$$

Here the x^2 on each side of the equation will destroy each other, and then by transposition, we get

$$4x=16-4=12,$$
$$\therefore\ x=\tfrac{12}{4}=3.$$

4. $$3x-2=\frac{6x^2+1}{2x-3}+8.$$

Adding 2 to each side of the equation, we get

$$3x=\frac{6x^2+1}{2x-3}+10.$$

Multiplying every term by $2x-3$, we get

$$6x^2-9x=6x^2+1+20x-30.$$

Here the $6x^2$ on each side of the equation will destroy each other, and then by transposition, we get

$$-9x-20x=1-30,$$
$$\therefore -29x=-29,$$
$$\therefore -x=-1,$$

And changing the signs, $x=1$.

5. $$\frac{3x-2}{x+1}+\frac{9x-21}{x^2-1}=3.$$

Here it will be observed that the denominator $x^2-1=(x+1)\times(x-1)$; therefore, by multiplying every term of the equation by this quantity, we shall clear it of fractions;

$$\therefore (3x-2)(x-1)+9x-21=3(x^2-1),$$
$$\therefore 3x^2-5x+2+9x-21=3x^2-3.$$

Here the $3x^2$ on each side of the equation destroy each other; and then

$$4x-19=-3,$$
$$\therefore 4x=19-3=16,$$
$$\therefore x=\tfrac{16}{4}=4.$$

6. $$\frac{3}{x+1}+\frac{4}{2x+2}+\frac{3x-3}{x^2-1}=1.$$

By obvious forms of reduction, we get

$$\frac{3}{x+1}+\frac{2}{x+1}+\frac{3(x-1)}{(x+1)(x-1)}=1,$$
$$\therefore \frac{3}{x+1}+\frac{2}{x+1}+\frac{3}{x+1}=1.$$

Multiplying every term of the equation by $x+1$, we get

$$3+2+3=x+1$$
$$\therefore x=7.$$

8. $$\frac{5x^2+25x}{x+9}=x+16.$$

Multiplying both sides of the equation by $x+9$, we get

$$5x^2+25x=(x+16)\ (x+9),$$
$$\therefore\ 5x^2+25x=x^2+25x+144.$$

Here the $25x$ on each side of the equation destroy each other; and then we find by transposition,

$$4x^2=144,$$
$$\therefore x^2=\frac{144}{4}=36.$$

Taking the square root of each side of the equation, we have

$$\therefore\ x=\sqrt{36}=6 \text{ or } -6.$$

Here it must be observed that the square root of 36 is either $+6$ or -6, because $+6\times+6=36$, and also $-6\times-6=36$.

10. $$\sqrt{3x+4}=5.$$

Squaring each side of the equation, we get

$$3x+4=25$$
$$\therefore\ 3x=25-4=21$$
$$\therefore\ x=\tfrac{21}{3}=7.$$

11. $$\sqrt{4x^2+16}=5.$$

Squaring each side of the equation, we get

$$4x^2+16=25,$$
$$\therefore\ 4x^2=25-16=9,$$
$$\therefore\ x^2=\tfrac{9}{4},$$

Taking the square root of each side,

$$x=\sqrt{\tfrac{9}{4}}=\tfrac{3}{2} \text{ or } -\tfrac{3}{2}.$$

13. $$\sqrt{x^2+8x-5}=x+3.$$

Squaring each side of the equation, we find

$$x^2+8x-5=(x+3)^2=x^2+6x+9.$$

Here the x^2 on each side of the equation destroy each other; and then we find, by transposition,

$$8x-6x=9+5,$$
$$\therefore\ 2x=14, \text{ and } x=\tfrac{14}{2}=7.$$

14. $$\sqrt{2x^2+4x-5}=x+2$$

Squaring each side, in order to clear the equation of the square root,

$$2x^2+4x-5=x^2+4x+4.$$

Taking all the x's to the left-hand side of the equation, and all the known quantities to the right,

$$2x^2+4x-x^2-4x=4+5,$$
$$\therefore\ x^2=9,$$
$$\therefore\ x=\sqrt{9}=3 \text{ or } -3.$$

16. $$\sqrt{x}-4=\frac{\sqrt{x}-2}{3}-\frac{\sqrt{x}-5}{12}.$$

Multiplying every term of the equation by 12, to clear it of fractions,

$$12\sqrt{x}-48=4(\sqrt{x}-2)-(\sqrt{x}-5),$$
$$\therefore\ 12\sqrt{x}-48=4\sqrt{x}-8-\sqrt{x}+5,$$
$$12\sqrt{x}=3\sqrt{x}-3+48,$$
$$9\sqrt{x}=45,$$
$$\therefore\ \sqrt{x}=\tfrac{45}{9}=5,$$
$$\therefore\ x=25.$$

19. $$x=\frac{bx+a}{c}.$$

Multiplying each side of the equation by c, to clear it of fractions,

$$cx=bx+a,$$

Taking bx from each side,

$$cx-bx=a.$$

Collecting the x's, as explained in Art. **32.**,

$$(c-b)\,x=a.$$

Dividing each side by $c-b$, we have

$$x=\frac{a}{c-b}.$$

20. $$\frac{a}{x}+\frac{b}{x}=d.$$

Multiplying each side of the equation by x,

$$a+b=dx, \text{ or } dx=a+b,$$

$$\therefore\ x=\frac{a+b}{d}.$$

21. $$(x+a)^2-a^2=b+2ax.$$

Squaring the quantity $x+a$, we have

$$x^2+2ax+a^2-a^2=b+2ax,$$

$$\therefore\ x^2+2ax=b+2ax,$$

$$\therefore\ x^2=b,$$

$$\therefore\ x=\sqrt{b}.$$

22. $$\frac{x}{a}+\frac{x}{b}=1.$$

Multiplying every term of the equation by ab,

$$bx+ax=ab,$$

$$\therefore\ (a+b)\,x=ab,$$

$$\therefore\ x=\frac{ab}{a+b}.$$

37. EXAMPLES IN EQUATIONS WITH TWO UNKNOWN QUANTITIES.

2. $$3x+2y=7 \quad . \quad . \quad (1.)$$
$$4x+5y=14 \quad . \quad . \quad (2.)$$

In order to make the number of x's the same in each equation, we multiply the first by 4, and the second by 3; then

$$12x+\ 8y=28,$$
$$12x+15y=42.$$

Subtracting the first of these equations from the second, in order to eliminate, or destroy, the x's, we have

$$7y=14,$$
$$\therefore\ y=\tfrac{14}{7}=2.$$

Substituting this 2 for y in equation (1.), we have

$$3x+2\times 2=7,$$
$$\therefore\ 3x=7-4=3,$$
$$\therefore\ x=1.$$

Or thus. The value of x may be found by the same method that we employed for finding the value of y; thus :—

Multiplying eq. (1.) by 5, and eq. (2.) by 2, so as to make the number of y's the same in each equation, we get

$$15x+10y=35,$$
$$8x+10y=28.$$

Subtracting the latter equation from the former, we get

$$7x=7,\ \therefore\ x=1, \text{ as before.}$$

To verify the results, or to show that the values of x and y have been correctly found, we have, by substituting 1 for x and 2 for y in the expressions (1.) and (2.),

$$3x+2y=3\times 1+2\times 2=7;\ \text{and}\ 4x+5y=4\times 1+5\times 2=14.$$

3. $$2x-3y=4 \quad . \quad . \quad (1.)$$
$$3x+2y=32 \quad . \quad . \quad (2.)$$

Multiplying eq. (1.) by 3, and eq. (2.) by 2, we get

$$6x-9y=12,$$
$$6x+4y=64.$$

Subtracting the former from the latter, we have

$$4y+9y=64-12,$$
$$\therefore\ 13y=52$$
$$\therefore\ y=\tfrac{52}{13}=4.$$

Substituting 4 for y in eq. (1.), we have

$$2x-3\times4=4,$$
$$\therefore\ 2x=4+12=16,$$
$$\therefore\ x=\tfrac{16}{2}=8.$$

To verify the results. Putting 8 for x and 4 for y in the expressions (1.) and (2.), we have

$$2x-3y=2\times8-3\times4=4, \text{ and } 3x+2y=3\times8+2\times4=32.$$

4. $$7x-4y=31 \quad . \quad . \quad . \quad (1.)$$
$$x+4y=9 \quad . \quad . \quad . \quad (2.)$$

Adding these equations together, we find

$$7x+x=40,$$
$$\therefore\ 8x=40,$$
$$\therefore\ x=\tfrac{40}{8}=5.$$

Substituting 5 for x in eq. (2.), we find

$$5+4y=9,$$
$$\therefore\ 4y=9-5=4,$$
$$\therefore\ y=\tfrac{4}{4}=1.$$

Art. 38.

2. $$x-y=1 \quad . \quad . \quad . \quad (1.)$$
$$7x-6y=15 \quad . \quad . \quad . \quad (2.)$$

Find the value of x from eq. (1.)

$$x=1+y \quad . \quad . \quad (3.)$$

Find the value of x from eq. (2.)

$$x=\frac{15+6y}{7}.$$

These two values for the same thing, must be equal to each other;

$$\therefore\ 1+y=\frac{15+6y}{7}.$$

Multiplying each side of the equation by 7;

$$7+7y=15+6y,$$
$$\therefore\ 7y-6y=15-7,$$
$$\therefore\ y=8.$$

Substituting this value of y in eq. (3.), we get

$$x=1+y=1+8=9.$$

Or thus. By the method given in Art. **37.** Multiplying eq. (1.) by 7, we get

$$7x-7y=7$$
$$7x-6y=15.$$

Subtracting the former from the latter,

$$7y-6y=8,$$
$$\therefore\ y=8, \text{ as before.}$$

And so on for the value of x.

3. $$3x+y=20 \quad . \quad . \quad . \quad (1.)$$
$$x-2y=2 \quad . \quad . \quad . \quad (2.)$$

Find the value of x from eq. (1.)

$$x=\frac{20-y}{3}.$$

Find the value of x from eq. (2.),

$$x=2+2y \quad . \quad . \quad (3.)$$

Therefore, we have by equality,

$$2+2y=\frac{20-y}{3}.$$

Multiplying each side of this equation by 3,

$$6+6y=20-y,$$
$$\therefore\ 7y=20-6=14,$$
$$\therefore\ y=\tfrac{14}{7}=2.$$

Substituting this value of y in eq. (3.), we get

$$x=2+2y=2+2\times2=6.$$

Or thus. By the method given in Art. 37. Multiplying eq. (1.) by 2, and writing the other eq. down,

$$6x+2y=40$$
$$x-2y=2$$

Adding these equations together, we find

$$7x=42,$$
$$\therefore\ x=\tfrac{42}{7}=6, \text{ as before.}$$

Again, multiplying eq. (2.) by 3,

$$3x+y=20$$
$$3x-6y=6.$$

Subtracting the latter from the former, we find

$$7y=14$$
$$\therefore\ y=\tfrac{14}{7}=2, \text{ as before.}$$

4. $$\left.\begin{array}{l}\dfrac{x}{2}-3y=3\\[2ex] \dfrac{x}{3}+2y=6\end{array}\right\}$$

Multiplying the first equation by 2, and the second by 3, to clear them of fractions,

$$x-6y=6\ \ldots\ (1.)$$
$$x+6y=18\ \ldots\ (2.)$$

From eq. (1.), we find

$$x=6+6y\ \ldots\ (3.)$$

And from eq. (2.), we find

$$x=18-6y.$$

Therefore we have by equality,

$$6+6y=18-6y,$$
$$\therefore\ 12y=12,$$
$$\therefore\ y=1.$$

Substituting this value of y in eq. (3.), we find

$$x=6+6y=6+6\times1=12.$$

Or thus. By method Art. **37.** Adding eq. (1.) and eq. (2.); then

$$2x=18+6=24,$$
$$\therefore\ x=\tfrac{24}{2}=12, \text{ as before.}$$

Subtracting eq. (1.) from eq. (2.); then

$$12y=18-6=12$$
$$\therefore\ y=\tfrac{12}{12}=1, \text{ as before.}$$

5. $$\left.\begin{aligned}\frac{x}{5}+3y&=11\\ x+2y&=16\end{aligned}\right\}$$

Multiplying the first equation by 5 to clear it of fractions, and writing down the other equation,

$$x+15y=55 \ .\ .\ .\ (1.)$$
$$x+\ \ 2y=16 \ .\ .\ .\ (2.)$$

Find the value of x from eq. (1.).

$$x=55-15y.$$

Find the value of x from eq. (2.)

$$x=16-2y \ .\ .\ .\ (3.)$$

Therefore by equality, we have

$$16-2y=55-15y,$$
$$\therefore\ 15y-2y=55-16,$$
$$\therefore\ 13y=39,$$
$$\therefore\ y=\tfrac{39}{13}=3.$$

Substituting this value of y in eq. (3.), we find

$$x=16-2y=16-2\times 3=10.$$

Or thus. By method Art. **34.** Subtracting eq. (2,) from eq. (1.); then

$$.\ 13y=39,$$
$$\therefore\ y=\tfrac{39}{13}=3, \text{ as before.}$$

Multiplying eq. (1.) by 2, and eq. (2.) by 15, we get

$$2x+30y=110$$
$$15x+30y=240.$$

Subtracting the former from the latter, we find

$$13x=130$$
$$\therefore x=10, \text{ as before.}$$

Art. 39.

2.
$$\left.\begin{array}{l}\frac{x}{2}+4y=10+y\\ 3x+7y=38\end{array}\right\}$$

Reduce the first equation, that is, clear it of fractions, and bring all the unknown quantities to the right hand side of the equation; then

$$x+6y=20\ .\ .\ .\ (1.)$$
$$3x+7y=38\ .\ .\ .\ (2.)$$

Find the value of x from eq. (1.)

$$x=20-6y\ .\ .\ .\ (3.)$$

Substitute this value of x in eq. (2.); then

$$3\,(20-6y)+7y=38,$$
$$\therefore\ 60-18y+7y=38,$$
$$\therefore\ 11y=22,\ \therefore\ y=\tfrac{22}{11}=2.$$

Substituting this value of y in eq. (3.),

$$x=20-6y=20-6\times 2=8.$$

Or thus. By method Art. **37.** Multiplying eq. (1.) by 3, and writing the other equation down,

$$3x+18y=60$$
$$3x+\ 7y=38.$$

Subtracting the latter equation from the former, we find

$$11y=22$$
$$\therefore\ y=\tfrac{22}{11}=2, \text{ as before.}$$

Substituting this value of y in eq. (1.), we get

$$x+6\times2=20$$
$$\therefore\ x=20-12=8, \text{ as before.}$$

Or thus. By method Art. **38.** From eq. (1.), we find

$$x=20-6y.$$

And from eq. (2.), we find

$$x=\frac{38-7y}{3}.$$
$$\therefore\ \frac{38-7y}{3}=20-6y,$$
$$\therefore\ 38-7y=60-18y,$$
$$\therefore\ 18y-7y=60-38,$$
$$\therefore\ 11y=22, \text{ and } y=2, \text{ as before.}$$

And so on, as in the last method, for the value of x.

3. $$\left.\begin{aligned}\frac{x}{5}-2y+4&=1\\ \frac{x}{3}+\frac{6y}{9}\quad&=7\end{aligned}\right\}$$

Reducing these equations, we get

$$x-10y=-15\ \ldots\ (1.)$$
$$x+\ 2y=\ \ 21\ \ldots\ (2.)$$

Subtract eq. (1.) from eq. (2.) then

$$2y+10y=21+15,$$
$$\therefore\ 12y=36,\ \therefore\ y=3.$$

To find x, multiply eq. (2.) by 5; then

$$x-10y=-15$$
$$5x+10y=\ 105.$$

Adding these equations together, we get

$$5x+x=105-15,$$
$$\therefore\ 6x=90;\ \therefore\ x=\tfrac{90}{6}=15.$$

4. $$\left.\begin{aligned}\frac{x+1}{2}+3y&=24\\ 5x+\frac{2y+1}{3}&=30\end{aligned}\right\}$$

To reduce these equations; multiply the first by 2, and transpose the constant; multiply the second by 3, and transpose the constant: the equations thus reduced become

$$x+6y=47 \quad . \; . \; . \; (1.)$$
$$15x+2y=89 \quad . \; . \; . \; (2.)$$

Find the value of x from eq. (1.)

$$x=47-6y \quad . \; . \; . \; (3.)$$

Substitute this value of x in eq. (2.); then

$$15\,(47-6y)\; +2y=89,$$
$$\therefore\; 705-90y+2y=89,$$
$$\therefore\; 88y=616,$$
$$\therefore\; y=\tfrac{616}{88}=7.$$

Substituting this value of y in eq. (3.),

$$x=47-6y=47-6\times 7=5.$$

Or thus. By method Art. **37.** Multiplying eq. (1.) by 15, and writing the other equation down,

$$15x+90y=705$$
$$15x+\;\;2y=\;\;89.$$

Subtracting the latter equation from the former,

$$88y=616;\; \therefore\; y=7, \text{ as before.}$$

To find x, multiply eq. (2.) by 3, and write the other down.

$$x+6y=47$$
$$45x+6y=267.$$

Subtracting the former equation from the latter,

$$44x=220$$
$$\therefore\; x=\tfrac{220}{44}=5, \text{ as before.}$$

Or thus. By method Art. **38.** From eq. (1.), we find

$$x=47-6y \quad . \; . \; . \; (3.)$$

And from eq. (2.), we find

$$x=\frac{89-2y}{15}.$$
$$\therefore\; \frac{89-2y}{15}=47-6y.$$

Multiplying both sides of this equation by 15, we get

$$89-2y=705-90y,$$
$$\therefore\ 90y-2y=704-89,$$
$$\therefore\ 88y=616;\ \therefore\ y=7, \text{ as before.}$$

Substituting this value of y in eq. (3.),

$$x=47-6y=47-6\times7=5, \text{ as before.}$$

40. PROBLEMS IN EQUATIONS OF TWO UNKNOWN QUANTITIES.

1. $$7x+3y=17 \quad . \ . \ . \ (1.)$$
$$5x+2y=12 \quad . \ . \ . \ (2.)$$

Multiplying eq. (1.) by 5, and eq. (2.) by 7, we find

$$35x+15y=85$$
$$35x+14y=84.$$

Subtracting the latter equation from the former, we find

$$y=1.$$

Multiplying eq. (1.) by 2, and eq. (2.) by 3, we find

$$14x+6y=34$$
$$15x+6y=36.$$

Subtracting the former equation from the latter, we find

$$x=2.$$

2. Let x=cost of each sheep, in shillings,
and y= „ lamb, „
∴ Cost of the 1st purchase in shillings$=5x+3y$,
and „ 2nd „ „ $=3x+5y$.

But the first is equal to £3 12*s.*=72 shillings, and the second is equal to £2 16*s.*=56 shillings; hence we have the two following equations:

$$5x+3y=72 \quad . \ . \ . \ (1.)$$
$$3x+5y=56 \quad . \ . \ . \ (2.)$$

Multiplying eq. (1.) by 3, and eq. (2.) by 5, we get

$$15x+\ 9y=216$$
$$15x+25y=280$$

Subtracting the former equation from the latter,

$$16y=64,$$

$$\therefore\ y=\tfrac{64}{16}=4s.,\text{ the price of each lamb.}$$

Multiplying eq. (1.) by 5, and eq. (2.) by 3, we get

$$25x+15y=360$$
$$9x+15y=168$$

Subtracting the latter equation from the former,

$$16x=192,$$

$$\therefore\ x=\tfrac{192}{16}=12s.,\text{ the price of each sheep.}$$

3. Let $x=$ the No. lbs of tea,
and $y=$ „ „ coffee.

Cost of the tea and coffee in the 1st case, in $s.=5x+2y$
„ „ 2nd „ „ $=6x+3y$.

But the first cost is equal to 22*s.*, and the second to 30*s.* Hence we have the two following equations:

$$5x+2y=22\ \ldots\ldots\ (1.)$$
$$6x+3y=30\ \ldots\ldots\ (2.)$$

To find y. Multiplying eq. (1.) by 6, and eq. (2.) by 5, we get

$$30x+12y=132$$
$$30x+15y=150$$

Subtracting the former equation from the latter,

$$3y=18,$$

$$\therefore\ y=\tfrac{18}{3}=6\text{ lbs., wt. of the tea.}$$

To find x. Multiplying eq. (1.) by 3, and eq. (2.) by 2, we get

$$15x+6y=66$$
$$12x+6y=60.$$

Subtracting the latter equation from the former,

$$3x=6$$

$$\therefore\ x=2\text{ lbs., wt. of the coffee.}$$

4. $$\left.\begin{array}{l} x+2=y-2 \\ y+2=3\,(x-2) \end{array}\right\}$$

These equations reduced become,

$$x-y=-4 \quad . \quad . \quad . \quad . \quad (1.)$$
$$y-3x=-8 \quad . \quad . \quad . \quad . \quad (2.)$$

Adding these equations together, we find

$$-2x=-12,$$
$$\therefore\ x=\tfrac{12}{2}=6$$

Substituting this value of x in eq. (2.), we find

$$y-3\times6=-8,$$
$$\therefore\ y=18-8=10.$$

5. Let x = the no. of James's marbles
and y = „ Thomas's „
$x+6$ = James's marbles increased by 6
$2y$ = twice Thomas's marbles.

But these numbers, by the question, are equal,

$$\therefore\ x+6=2y \quad . \quad . \quad . \quad . \quad (1.)$$

Again, to obtain another equation, we have

$y+9$ = Thomas's marbles increased by 9
$2x$ = twice James's marbles.

But these numbers, by the question, are equal,

$$\therefore\ 2x=y+9 \quad . \quad . \quad . \quad . \quad (2.)$$

From eq. (1.), we find

$$x=2y-6 \quad . \quad . \quad . \quad . \quad (3.)$$

Substituting this value of x in eq. (2.), we get

$$2\,(2y-6)=y+9,$$
$$\therefore\ 4y-12=y+9,$$
$$\therefore\ 4y-y=9+12,$$
$$\therefore\ 3y=21, \text{ and } y=7.$$

Substituting this value of y in eq. (3.), we find

$$x=2y-6=2\times7-6=8.$$

6. Let x=the father's age,
and y=the son's age.
$x+5$=the father's age 5 years hence
$y+5$= „ son's „ „

But, by the question, the latter is one-fourth the former

$$\therefore\ y+5=\tfrac{1}{4}(x+5)$$

Clearing this equation of fractions, &c., we get

$$x-4y=15 \quad . \quad . \quad . \quad . \quad (1.)$$

Again, to obtain another equation, we have

$x+15$=the father's age 15 years hence,
$y+15$= „ son's „ „

But, by the question, the latter is two-fifths of the former,

$$\therefore\ y+15=\tfrac{2}{5}(x+15)$$

Clearing this equation of fractions, &c., we get

$$2x-5y=45 \quad . \quad . \quad . \quad . \quad (2.)$$

From eq. (1.) we find

$$x=15+4y \quad . \quad . \quad . \quad . \quad (3.)$$

Substituting this value of x in eq. (2.), we have

$$2(15+4y)-5y=45$$
$$\therefore\ 30+8y-5y=45$$
$$\therefore\ 3y=15, \text{ and } y=5, \text{ the son's age.}$$

Substituting this value of y in eq. (3.), we find

$$x=15+4y=15+4\times5=35, \text{ the father's age.}$$

8. Let x=no. lbs. of sugar at 4*d.*
and y= „ „ 8*d.*

Here we may obtain two independent values for the cost of the mixture.

1st, Cost of the mixture, in pence$=4x+8y+6\times25$
2nd, „ „ „ $=7\times70=490$

$$\therefore\ 4x+8y+6\times25=490$$
$$\therefore\ 4x+8y=340 \quad . \quad . \quad . \quad . \quad (1.)$$

Now the sum of the weights of the different sorts of sugar must make up 70 lbs.; hence we obtain another equation,

$$x+y+25=70,$$

$$\therefore x+y=45 \quad . \quad . \quad . \quad . \quad (2.)$$

Subtracting 4 times eq. (2.) from eq. (1.) we obtain

$$4y=160$$

$$\therefore y=\tfrac{160}{4}=40 \text{ lbs. sugar at } 8d.$$

Substituting this value of y in eq. (2.), we have

$$x+40=45$$

$$\therefore x=5 \text{ lbs., sugar at } 4d.$$

9. Let x=Peter's age,
and y=John's age.

Here, from the data of the question, we readily derive the two following equations:

$$x+38=4y,$$

$$\text{and } y+38=5x.$$

Reducing these equations, we find

$$4y-x=38 \quad . \quad . \quad . \quad . \quad (1.)$$

$$5x-y=38 \quad . \quad . \quad . \quad . \quad (2.)$$

Adding eq. (1.), and 4 times eq. (2.) together, we obtain

$$19x=190,$$

$$\therefore x=\tfrac{190}{19}=10, \text{ Peter's age.}$$

Again, adding eq. (2.), and 5 times eq. (1.) together, we obtain

$$19y=228,$$

$$\therefore y=\tfrac{228}{19}=12, \text{ John's age.}$$

10. $$10x+y=2(x+y) \quad . \quad . \quad . \quad . \quad (1.)$$

$$10x+y+9=\frac{10y+x}{3} \quad . \quad . \quad . \quad . \quad (2.)$$

Reducing eq. (1.), we have

$$10x+y=2x+2y,$$

$$\therefore 10x-2x=2y-y,$$

$$\therefore 8x=y, \text{ or } y=8x.$$

Now putting $8x$ for y in eq. (2.), we have

$$10x+8x+9=\frac{10\times 8x+x}{3}$$

$$=\frac{81x}{3}=27x,$$

$$\therefore\ 9=9x;\ \therefore\ x=1;$$

$$\text{and } y=8x=8\times 1=8.$$

11. Let x=the digit in the ten's place,
and y= „ „ unit's „
$\therefore$ $10x+y$=the number,
$10y+x$=the number with its digits inverted,
and $12(y-x)$=12 times the difference of its digits.

Here it must be observed that the unit's digit is greater than the ten's digit, because the number is increased by the inversion of the digits: hence we put $y-x$ for the difference of the digits.

From the conditions of the question we readily derive the two following equations:

$$10x+y=12(y-x)\ldots(1.)$$

$$10x+y+36=10y+x\ldots\ldots(2.)$$

Reducing eq. (1.), we have

$$10x+y=12y-12x,$$

$$\therefore\ 22x=11y,\ \therefore\ y=2x.$$

Now putting $2x$ for y in eq. (2.), we get

$$10x+2x+36=10\times 2x+x,$$

$$\therefore\ 36=9x,\ \therefore\ x=\tfrac{36}{9}=4;$$

$$\text{and } y=2x=2\times 4=8.$$

$\therefore$ The number required$=10x+y=40+8=48$.

12. Let x=A's capital at first, in pounds,
and y=B's „ „ „
A's capital after gaining £40$=x+40$,
„ „ „ losing £80$=x+40-80=x-40$,
B's „ „ gaining £70$=y+70$.

Hence the conditions of the question give us the two following equations:

$$x+40=2y \text{ ....(1.)}$$
$$y+70=6(x-40) \text{ ....(2.)}$$

From eq. (1.), we find

$$x=2y-40 \quad \text{....(3.)}$$

Substituting this value of x in eq. (2.), we have

$$y+70=6(2y-40-40),$$
$$\therefore\ y+70=12y-480,$$
$$\therefore\quad 550=11y,\ \therefore\ y=\tfrac{550}{11}=\pounds 50, \text{ A's};$$

and from eq. (3.), we find

$$x=2y-40=2\times 50-40=\pounds 60, \text{ B's}.$$

13. Let x=the greater,
and y=the less.

Then from the conditions of the question we derive the two following equations:

$$x+y=2 \text{ ... (1.)}$$
$$x-y=1 \text{ ... (2.)}$$

Adding these equations together, we get

$$2x=3,\ \therefore\ x=\tfrac{3}{2}=1\tfrac{1}{2}, \text{ the greater.}$$

Subtracting eq. (2.) from eq. (1.), we get

$$2y=1,\ \therefore\ y=\tfrac{1}{2}, \text{ the less.}$$

42. EXAMPLES IN QUADRATIC EQUATIONS.

3. $$x^2+8x=33.$$

Here $\frac{8}{2}=4$, and $4^2=16$, $\therefore$ to *complete the square* we add 16 to each side of the equation.

$$\therefore\ x^2+8x+16=33+16=49,$$
$$\text{or } (x+4)^2=49.$$

Taking the square root of each side,

$$x+4=7 \text{ or } -7.$$

Here these two values of the root give us two equations for finding the values of x, that is,

$$x+4=7, \text{ or } x+4=-7,$$
$$\therefore\ x=3, \text{ or } \quad x=-11.$$

Hence we write, $x=3$ or -11.

4. $$x^2+10x=24$$

Here $\frac{10}{2}=5$, and $5^2=25$, $\therefore$ adding 25 to each side,

$$x^2+10x+25=24+25=49$$
$$\text{or } (x+5)^2=49$$
$$\therefore\ x+5=7 \text{ or } -7, \text{ that is,}$$
$$x+5=7, \text{ or } x+5=-7,$$
$$x=2, \text{ or } \quad x=-12.$$

Hence we write, $x=2$ or -12.

5. $$x^2+12x=85.$$

Here $\frac{12}{2}=6$, and $6^2=36$, $\therefore$ adding 36 to each side.

$$x^2+12x+36=85+36=121,$$
$$\text{or } (x+6)^2=121,$$
$$\therefore\ x+6=11 \text{ or } -11, \text{ that is,}$$
$$x+6=11, \text{ or } x+6=-11,$$
$$\therefore\ x=5, \text{ or } \quad x=-17.$$

Hence we write, $x=5$ or -17.

6. $$x^2-6x=7.$$

Here $\frac{6}{2}=3$, and $3^2=9$, $\therefore$ adding 9 to each side,

$$x^2-6x+9=7+9=16,$$
$$\text{or } (x-3)^2=16, \text{ or } (\pm 4)^2,$$
$$\therefore\ x-3=4 \text{ or } -4, \text{ that is,}$$
$$x-3=4, \text{ or } x-3=-4,$$
$$\therefore\ x=7, \text{ or } \quad x=-4+3=-1.$$

Hence we write, $x=7$ or -1.

7. $$x^2-16x=80.$$

Here $\frac{16}{2}=8$, and $8^2=64$, ∴ adding 64 to each side,

$$x^2-16x+64=80+64=144,$$

or $(x-8)^2=144$, or $(\pm 12)^2$,

∴ $x-8=12$, or -12, that is,

$x-8=12$, or $x-8=-12$,

∴ $x=20$, or $x=-12+8=-4$

8. $$x^2-2x=3.$$

Here $\frac{2}{2}=1$, and $1^2=1$, ∴ adding 1 to each side,

$$x^2-2x+1=3+1=4,$$

or $(x-1)^2=4$, or $(\pm 2)^2$,

∴ $x-1=2$, or -2, that is,

$x-1=2$, or $x-1=-2$.

∴ $x=3$, or $x=-2+1=-1$.

10. $$x^2+3x=40.$$

Here the half of 3 is $\frac{3}{2}$, and $(\frac{3}{2})^2=\frac{9}{4}$, ∴ adding $\frac{9}{4}$ to each side,

$$x^2+3x+\tfrac{9}{4}=40+\tfrac{9}{4}=\tfrac{169}{4},$$

or $(x+\frac{3}{2})^2=\frac{169}{4}$, or $(\pm\frac{13}{2})^2$;

Taking the square root of each side

$x+\frac{3}{2}=\frac{13}{2}$ or $-\frac{13}{2}$, that is,

$x+\frac{3}{2}=\frac{13}{2}$, or $x+\frac{3}{2}=-\frac{13}{2}$

∴ $x=5$, or $x=-\frac{13}{2}-\frac{3}{2}=-8$.

11. $$x^2+7x=30.$$

Here $(\frac{7}{2})^2=\frac{49}{4}$, ∴ adding $\frac{49}{4}$ to each side,

$$x^2+7x+\tfrac{49}{4}=30+\tfrac{49}{4}=\tfrac{169}{4},$$

or $(x+\frac{7}{2})^2=\frac{169}{4}$, or $(\pm\frac{13}{2})^2$;

Taking the square root of each side,

∴ $x+\frac{7}{2}=\frac{13}{2}$, or $-\frac{13}{2}$, that is,

$x+\frac{7}{2}=\frac{13}{2}$, or $x+\frac{7}{2}=-\frac{13}{2}$,

∴ $x=3$, or $x=-10$.

12. $x^2+9x=10.$

Here $(\frac{9}{2})^2=\frac{81}{4}$, $\therefore$ adding $\frac{81}{4}$ to each side,

$$x^2+9x+\frac{81}{4}=10+\frac{81}{4}=\frac{121}{4},$$

$$\text{or } (x+\frac{9}{2})^2=\frac{121}{4}, \text{ or } (\pm\frac{11}{2})^2;$$

Taking the square root of each side,

$$x+\frac{9}{2}=\frac{11}{2}, \text{ or} -\frac{11}{2}, \text{ that is,}$$

$$x+\frac{9}{2}=\frac{11}{2}, \text{ or } x+\frac{9}{2}=-\frac{11}{2},$$

$$\therefore x=1, \text{ or } \quad x=-10.$$

13. $x^2-7x=8.$

Here $(\frac{7}{2})^2=\frac{49}{4}$, $\therefore$ adding $\frac{49}{4}$ to each side,

$$x^2-7x+\frac{49}{4}=8+\frac{49}{4}=\frac{81}{4},$$

$$\text{or } (x-\frac{7}{2})^2=\frac{81}{4}, \text{ or } (\pm\frac{9}{2})^2;$$

Taking the square root of each side,

$$x-\frac{7}{2}=\frac{9}{2}, \text{ or } -\frac{9}{2}, \text{ that is,}$$

$$x-\frac{7}{2}=\frac{9}{2}, \text{ or } x-\frac{7}{2}=-\frac{9}{2},$$

$$\therefore x=8, \text{ or } \quad x=-1.$$

14. $x^2-13x=140.$

Here $(\frac{13}{2})^2=\frac{169}{4}$, which must be added to *complete the square.*

$$x^2-13x+\frac{169}{4}=140+\frac{169}{4}=\frac{729}{4},$$

$$\text{or } (x-\frac{13}{2})^2=\frac{729}{4} \text{ or } (\pm\frac{27}{2})^2;$$

Taking the square root of each side,

$$x-\frac{13}{2}=\frac{27}{2}, \text{ or} -\frac{27}{2}, \text{ that is,}$$

$$x-\frac{13}{2}=\frac{27}{2}, \text{ or } x-\frac{13}{2}=-\frac{27}{2},$$

$$\therefore x=20, \text{ or } x=-7.$$

16. $4x^2+6x=54.$

Dividing by 4, the coefficient of x^2, we find

$$x^2+\frac{3}{2}x=\frac{27}{2}$$

Here, to complete the square, we have the half of $\frac{3}{2}=\frac{3}{4}$, and $(\frac{3}{4})^2=\frac{9}{16}$; hence we add $\frac{9}{16}$ to each side,

$$x^2+\frac{3}{2}x+\frac{9}{16}=\frac{27}{2}+\frac{9}{16}=\frac{225}{16}$$

$$\text{or } (x+\frac{3}{4})^2=\frac{225}{16}, \text{ or } (\pm\frac{15}{4})^2;$$

Taking the square root of each side,

$$x+\tfrac{3}{4}=\tfrac{15}{4}, \text{ or} -\tfrac{15}{4},$$
$$\therefore\ x=\tfrac{15}{4}-\tfrac{3}{4}, \text{ or} -\tfrac{15}{4}-\tfrac{3}{4}$$
$$=3, \text{ or } -\tfrac{9}{2}.$$

17. $$8x^2-4x=24.$$

Dividing by 8, we find,

$$x^2-\tfrac{1}{2}x=3.$$

Here the half of $\frac{1}{2}=\frac{1}{4}$, and $(\frac{1}{4})^2=\frac{1}{16}$, therefore, adding $\frac{1}{16}$ to each side,

$$x^2-\tfrac{1}{2}x+\tfrac{1}{16}=3+\tfrac{1}{16}=\tfrac{49}{16},$$
$$\text{or } (x-\tfrac{1}{4})^2=\tfrac{49}{16}, \text{ or } (\pm\tfrac{7}{4})^2$$
$$\therefore\ x-\tfrac{1}{4}=\tfrac{7}{4}, \text{ or} -\tfrac{7}{4}, \text{ that is,}$$
$$x-\tfrac{1}{4}=\tfrac{7}{4}, \text{ or } x-\tfrac{1}{4}=-\tfrac{7}{4},$$
$$\therefore\ x=2, \text{ or } x=-\tfrac{6}{4}=-1\tfrac{1}{2}.$$

18. $$9x^2-2x=7.$$

Dividing by 9, we get

$$x^2-\tfrac{2}{9}x=\tfrac{7}{9}.$$

Here $\frac{1}{2}$ of $\frac{2}{9}=\frac{1}{9}$, and $(\frac{1}{9})^2=\frac{1}{81}$, $\therefore$ adding $\frac{1}{81}$ to each side,

$$x^2-\tfrac{2}{9}x+\tfrac{1}{81},=\tfrac{7}{9}+\tfrac{1}{81}=\tfrac{64}{81}$$
$$(\text{or } x-\tfrac{1}{9})^2=\tfrac{64}{81}, \text{ or } (\pm\tfrac{8}{9})^2.$$

Taking the square root of each side,

$$x-\tfrac{1}{9}=\tfrac{8}{9}, \text{ or } -\tfrac{8}{9}$$
$$\therefore\ x=1, \text{ or} -\tfrac{7}{9}$$

20. $$x^2+{\cdot}26x=8{\cdot}3.$$

Here $\frac{\cdot 26}{2}={\cdot}13$, and ${\cdot}13^2={\cdot}0169$, $\therefore$ adding this to each side,

$$x^2+{\cdot}26x+{\cdot}0169=8{\cdot}3+{\cdot}0169=8{\cdot}3169,$$
$$\textit{or } (x+{\cdot}13)^2=8{\cdot}3169;$$

Taking the square root of each side,

$$x+{\cdot}13=\pm\sqrt{8{\cdot}3169}=2{\cdot}883 \text{ or } -2{\cdot}883.$$
$$\therefore\ x=2{\cdot}883-{\cdot}13=2{\cdot}753, \textit{ or } -2{\cdot}883-{\cdot}13=-3{\cdot}013.$$

21. $$5x^2+3x=7{\cdot}25.$$

Dividing by 5, we find,

$$x^2+\tfrac{3}{5}x=\frac{7{\cdot}25}{5}=1{\cdot}45.$$

Here $\frac{3}{5}\div 2=\frac{3}{10}={\cdot}3$, and ${\cdot}3^2={\cdot}09$, ∴ adding this to each side,

$$x^2+\tfrac{3}{5}x+{\cdot}09=1{\cdot}45+{\cdot}09=1{\cdot}54,$$
$$\text{or } (x+{\cdot}3)^2=1{\cdot}54;$$

Taking the square root of each side,

$$x+{\cdot}3=\pm\sqrt{1{\cdot}54}=1{\cdot}2409 \text{ or } -1{\cdot}2409,$$
$$\therefore\ x=1{\cdot}2409-{\cdot}3={\cdot}9409,$$
$$\text{or } x=-1{\cdot}2409-{\cdot}3=-1{\cdot}5409.$$

22. $$x^2-4x=32.$$
$$\therefore\ x^2-4x+4=32+4=36,$$
$$\therefore\ (x-2)^2=36, \text{ or } (\pm 6)^2,$$
$$\therefore\ x-2=\ 6, \text{ or } -6,$$
$$\therefore\ x=6+2=8,$$
$$\text{or } x=-6+2=-4.$$

23. $$5x=\frac{8-x}{x}+7.$$

Multiplying each side of the equation by x,

$$5x^2=8-x+7x,$$

By transposition and collecting,

$$5x^2-6x=8,$$

Dividing every term of the equation by 5,

$$x^2-\tfrac{6}{5}x=\tfrac{8}{5}.$$

Here the half of $\frac{6}{5}$ is $\frac{3}{5}$, and $(\frac{3}{5})^2=\frac{9}{25}$. ∴ to *complete the square*, we add $\frac{9}{25}$ to each side of the equation.

$$x^2-\tfrac{6}{5}x+\tfrac{9}{25}=\tfrac{8}{5}+\tfrac{9}{25}=\tfrac{49}{25},$$
$$\text{or } (x-\tfrac{3}{5})^2=\tfrac{49}{25}, \text{ or } (\pm\tfrac{7}{5})^2,$$

Taking the square root of each side,

$$x-\tfrac{3}{5}=\pm\sqrt{\tfrac{49}{25}}=\pm\tfrac{7}{5}$$

$$\therefore\ x=+\tfrac{7}{5}+\tfrac{3}{5}=2$$

$$\text{or } x=-\tfrac{7}{5}+\tfrac{3}{5}=-\tfrac{4}{5}.$$

24. $$\frac{x}{x-2}-\frac{x-1}{x}=1+\frac{1}{x}.$$

By transposition, &c., we have

$$\frac{x}{x-2}=1+\frac{1}{x}+\frac{x-1}{x}=1+\frac{x}{x}=1+1=2,$$

Multiplying both sides of the equation by $x-2$,

$$x=2x-4,$$

$$\therefore\ x=4.$$

27. $$3x^4+12x^2=351.$$

Dividing every term by 3, we get

$$x^4+4x^2=117.$$

Here as the higher exponent is double the lower one, this equation may be solved by completing the square.

$$\therefore\ x^4+4x^2+4=121,$$

$$\text{or } (x^2+2)^2=121, \text{ or } (\pm 11)^2,$$

Taking the square root of each side,

$$x^2+2=11, \text{ or } -11,$$

$$\therefore\ x^2=9, \text{ or } -13.$$

Taking the square root,

$$x=\pm 3, \text{ or } \pm\sqrt{-13};$$

that is, we have $x=3$ or -3, which are the two *real* roots of the proposed equation; and $x=+\sqrt{-13}$ or $-\sqrt{-13}$, which are the two *imaginary* roots.

28. $$x^4-2x^2=224.$$

Here the exponent, 4, is double the exponent, 2, hence we have by completing the square,

$$x^4-2x^2+1=225,$$

$$\text{or } (x^2-1)^2=225, \text{ or } (+15)^2.$$

Taking the square root of each side,

$$x^2-1=15 \text{ or } -15$$

$$\therefore\ x^2=16 \text{ or } -14.$$

Taking the square root of each side,

$$x=+4, \text{ or } \pm\sqrt{-14};$$

that is, $x=4$ or -4, which are the two real roots, and $x=+\sqrt{-14}$ or $-\sqrt{-14}$, which are the imaginary roots.

29. $$x+10x^{\frac{1}{2}}=39.$$

Completing the square by adding 5^2 to each side,

$$x+10x^{\frac{1}{2}}+25=39+25=64,$$

$$\text{or } (x^{\frac{1}{2}}+5)^2=64, \text{ or } (\pm 8)^2,$$

$$\therefore\ x^{\frac{1}{2}}+5=8 \text{ or } -8,$$

$$\therefore\ x^{\frac{1}{2}}=3 \text{ or } -13.$$

From $x^{\frac{1}{2}}=3$, we get by squaring

$$x=9;$$

and from $x^{\frac{1}{2}}=-13$, we get by squaring

$$x=-13\times-13=169.$$

The proposed equation, therefore, gives two values for $x^{\frac{1}{2}}$, viz., 3 and -13, and *only* two values; so that the value $x=9$ is restricted to the value $x^{\frac{1}{2}}=3$, and the value $x=169$ is restricted to the value $x^{\frac{1}{2}}=-13$. As may be shown by substituting these values for x in the proposed equation; thus,

For $x=9$, and $x^{\frac{1}{2}}=3$, we have

$$9+10\times 3=39$$

For $x=169$, and $x^{\frac{1}{2}}=-13$, we have

$$169+10\times-13=39.$$

In like manner two values of x may be obtained for the equation given in example 26; thus

$$x+8x^{\frac{1}{2}}+16=36,$$

$$\therefore\ x^{\frac{1}{2}}+4=6 \text{ or } -6,$$

$$\therefore\ x^{\frac{1}{2}}=2 \text{ or } -10;$$

for $x^{\frac{1}{2}}=2$, we find $x=4$,

and for $x^{\frac{1}{2}}=-10$, we find $x=100$.

43. PROBLEMS PRODUCING QUADRATIC EQUATIONS OF ONE UNKNOWN QUANTITY.

1. $$x^2+x=42.$$

Completing the square by adding $(\frac{1}{2})^2$ to each side,

$$x^2+x+\tfrac{1}{4}=42+\tfrac{1}{4}=\tfrac{169}{4},$$
$$\text{or } (x+\tfrac{1}{2})^2=\tfrac{169}{4} \text{ or } (\pm\tfrac{13}{2})^2,$$
$$\therefore\ x+\tfrac{1}{2}=\tfrac{13}{2} \text{ or } -\tfrac{13}{2},$$
$$\therefore\ x=6 \text{ or} -7.$$

The latter value of x is rejected, being inconsistent with the nature of the problem.

2. $$x^2+2x=15.$$

Completing the square,

$$x^2+2x+1=15+1=16,$$
$$\text{or } (x+1)^2=16 \text{ or } (\pm 4)^2,$$
$$\therefore\ x+1=4 \text{ or } -4,$$
$$\therefore\ x=3 \text{ or } -5.$$

Taking $x=3$, the less; the greater $=3+2=5$; so that, in this case, the numbers will be 3 and 5.

But we have also $x=-5$; and the other number $=-5+2=-3$; so that, in this case, the numbers will be -3 and -5.

Both results, *algebraically* considered, are consistent with the conditions of the problem, for $5\times 3=15$, and also $-3\times -5=15$.

3. Let x=the number; then
$x+12$=the number increased by 12,
and x^2=the square of the number.

But from the data of the problem, we have

$$x^2=x+12,$$
$$\therefore\ x^2-x=12,$$

Completing the square by adding $(\frac{1}{2})^2$ to each side,

$$x^2-x+\tfrac{1}{4}=12+\tfrac{1}{4}=\tfrac{49}{4},$$
$$\text{or } (x-\tfrac{1}{2})^2=\tfrac{49}{4} \text{ or } (\pm\tfrac{7}{2})^2,$$
$$\therefore\ x-\tfrac{1}{2}=\tfrac{7}{2} \text{ or } -\tfrac{7}{2},$$
$$\therefore\ x=4 \text{ or } -3.$$

That both of these results answer the conditions of the problem, may be shown as follows:

Taking $x=4$; $4+12=4^2$ or 16

Taking $x=-3$; $-3+12=(-3)^2$ or 9.

4. Let x=number of oranges; then

$$\frac{40}{x}=\text{the cost price of each, in pence,}$$
$$x-4=\text{the number sold,}$$
$$\text{and } \frac{48}{x-4}=\text{the selling price of each, in pence.}$$

But as the selling price of each was 1*d.* more than the cost price, we have the following equation:

$$\frac{48}{x-4}-\frac{40}{x}=1.$$

Multiplying both sides of this equation, first by x, and then by $x-4$, we get

$$48x-40(x-4)=x(x-4),$$
$$\therefore\ 48x-40x+160=x^2-4x,$$
$$\therefore\ 8x+160=x^2-4x,$$
$$\therefore\ x^2-12x=160.$$

Completing the square by adding 6^2 to each side,

$$x^2-12x+36=160+36=196,$$
$$\text{or } (x-6)^2=196 \text{ or } (\pm 14)^2,$$
$$\therefore\ x-6=14 \text{ or } -14,$$
$$\therefore\ x=20 \text{ or } -8.$$

Where the latter value of x is rejected, being inconsistent with the nature of the problem.

5. Let x=the no. of yards; then

72=the cost price of whole, in shillings,

$\frac{72}{x}$=the cost price per yard, „ „

$\frac{7}{2}\times x$=the selling price of the whole „

∴ *The gain, in shillings*$=\frac{7}{2}x-72$.

But, from the question, the gain was equal to the cos 4 yards;

∴ *The gain, in shillings*$=\frac{72}{x}\times 4=\frac{288}{x}$

Hence, by equality, we get the following equation:

$$\frac{7}{2}x-72=\frac{288}{x}.$$

Multiplying every term by x,

$$\tfrac{7}{2}x^2-72x=288.$$

Dividing every term by $\frac{7}{2}$,

$$x^2-\tfrac{144}{7}=288\times\tfrac{2}{7}=\tfrac{576}{7}.$$

Here, in order to complete the square, we find the half $\frac{144}{7}=\frac{72}{7}$, and $(\frac{72}{7})^2=\frac{5184}{49}$.

$$\therefore\ x^2-\tfrac{144}{7}x+(\tfrac{72}{7})^2=\tfrac{576}{7}+\tfrac{5184}{49}=\tfrac{9216}{49},$$

$$\text{or } (x-\tfrac{72}{7})^2=\tfrac{9216}{49}, \text{ or } (\pm\tfrac{96}{7})^2,$$

$$\therefore\ x-\tfrac{72}{7}=\tfrac{96}{7}, \text{ or} -\tfrac{96}{7},$$

$$\therefore\ x=\tfrac{96}{7}+\tfrac{72}{7}=\tfrac{168}{7}=24,$$

$$\text{or } x=-\tfrac{96}{7}+\tfrac{72}{7}=-\tfrac{24}{7}.$$

The latter, or minus, value of x is neglected, not bei applicable to the problem proposed.

6. $$\frac{20}{x}-\frac{20}{x+3}=6.$$

Multiplying, first by x, and then by $x+3$, we find

$$20\,(x+3)-20x=6x\,(x+3),$$

$$\therefore\ 20x+60-20x=6x^2+18x,$$

$$\therefore\ 6x^2+18x=60,$$

$$\therefore\ x^2+3x=10.$$

Completing the square by adding $(\frac{3}{2})^2$ to each side,

$$x^2+3x+\tfrac{9}{4}=10+\tfrac{9}{4}=\tfrac{49}{4},$$
$$\text{or } (x+\tfrac{3}{2})^2=\tfrac{49}{4} \text{ or } (\pm\tfrac{7}{2})^2,$$
$$\therefore\ x+\tfrac{3}{2}=\tfrac{7}{2} \text{ or} -\tfrac{7}{2}$$
$$\therefore\ x=2 \text{ or} -5.$$

7. Let x=the no. miles travelled per hour.

21=the distance travelled, in miles;

$$\therefore\ \textit{no. hours in travelling}=\frac{21}{x}.$$

But, by the question, he took 4 hours more than the units in x, that is,

$$\textit{no. hours in travelling}=x+4.$$

Hence, by equality, we get the following equation:

$$x+4=\frac{21}{x},$$

$$\therefore\ x^2+4x=21.$$

Completing the square of this quadratic, we have

$$x^2+4x+4=25,$$
$$\text{or } (x+2)^2=25, \text{ or } (\pm 5)^2;$$

Taking the square root of each side,

$$x+2=5, \text{ or} -5,$$
$$\therefore\ x=3, \text{ or} -7.$$

Hence the rate per hour=3 miles, the minus value of x not being fairly applicable to the problem.

8. Let x=the no. miles per hour; then

$\frac{72}{x}$=no. hours on the journey.

$\frac{72}{x-1}$=no. hours on the journey, walking 1 mile per hour slower.

But, by the question, the former is 6 hours less than the latter; hence we have the following equation:

$$\frac{72}{x-1}-\frac{72}{x}=6,$$

$$\therefore \frac{12}{x-1}-\frac{12}{x}=1.$$

Multiplying, first by x, and then by $x-1$,

$$12x-12\,(x-1)=x\,(x-1),$$

$$\therefore\ 12x-12x\ +12=x^2-x,$$

$$\therefore x^2-x=12.$$

Completing the square by adding $(\frac{1}{2})^2$ to each side,

$$x^2-x+\tfrac{1}{4}=12+\tfrac{1}{4}=\tfrac{49}{4},$$

$$\text{or } (x-\tfrac{1}{2})^2=\tfrac{49}{4} \text{ or } (\pm\tfrac{7}{2})^2,$$

$$\therefore\ x-\tfrac{1}{2}=\tfrac{7}{2} \text{ or} -\tfrac{7}{2},$$

$$\therefore\ x=4 \text{ or} -3.$$

Hence the rate per hour$=4$ miles.

9. Let $x=$the one part; then

$9-x=$the other part.

Hence, from data of the problem, we derive the following equation:

$$x^2+\ (9-x)^2=53,$$

$$\therefore\ x^2+81-18x+x^2=53,$$

$$\therefore\ 2x^2-18x=53-81=-28$$

$$\therefore\ \ x^2-\ 9x=-\tfrac{28}{2}=-14.$$

Completing the square by adding $(\frac{9}{2})^2$ to each side,

$$x^2-9x+\tfrac{81}{4}=-14+\tfrac{81}{4}=\tfrac{25}{4},$$

$$\text{or}\qquad (x-\tfrac{9}{2})^2=\tfrac{25}{4} \text{ or } (\pm\tfrac{5}{2})^2,$$

$$\therefore x-\tfrac{9}{2}=\tfrac{5}{2} \text{ or } -\tfrac{5}{2}$$

$$\therefore\ x=7 \text{ or } 2.$$

The first part being 7, the other part will be 9—7 or 2; and the first being 2, the other part will be 9—2 or 7; so that the two values of x, in this case, give the two parts required in the problem.

10. Let x=the ten's digit; then
$7-x$=the unit's digit,
$x(7-x)$=the product of the digits,
And the number$=10x+7-x=9x+7$.

But, by the question, the former is 10 less than the latter; hence we derive the following equation:

$$9x+7-x(7-x)=10,$$
$$\therefore 9x+7-7x+x^2=10,$$
$$\therefore x^2+2x=10-7=3.$$

Completing the square by adding 1^2, or 1, to each side,

$$x^2+2x+1=3+1=4,$$
$$\text{or} \quad (x+1)^2=4 \text{ or } (\pm 2)^2,$$
$$\therefore x+1=2 \text{ or } -2,$$
$$\therefore x=1 \text{ or } -3.$$

Therefore, the ten's digit=1; and the unit's digit$=7-1=6$;
∴ the number required=16.

The minus value of x, in this case, does not admit of any feasible interpretation.

11. $$\frac{1}{x+6}+\frac{1}{x+\frac{3}{2}}=\frac{1}{x}$$

To clear the equation of fractions, multiply every term by $x(x+6)(x+\frac{3}{2})$; then

$$x(x+\tfrac{3}{2})+x(x+6)=(x+6)(x+\tfrac{3}{2}).$$

Performing the multiplications,

$$x^2+\tfrac{3}{2}x+x^2+6x=x^2+\tfrac{3}{2}x+6x+9.$$

Here cutting out the terms which destroy each other, we get

$$x^2=9, \therefore x=3.$$

12. Let x=the no. hours in which B alone can do it; then
$x-5$= „ „ A „ „
∴ The part done by B alone in 1 hour$=\frac{1}{x}$

The part done by A alone in one hour $=\frac{1}{x-5}$

„ A and B together „ $=\frac{1}{x}+\frac{1}{x-5}$

But, by the question, A and B together can perform the work in 6 hours, that is, they can perform $\frac{1}{6}$ of it in 1 hour;

$$\therefore\ \frac{1}{x}+\frac{1}{x-5}=\frac{1}{6}.$$

Multiplying every term by $6x\ (x-5)$, we have

$$6\ (x-5)\ +6x=x\ (x-5),$$
$$\therefore\ 6x-30+6x=x^2-5x,$$
$$\therefore\ x^2-17x=-30.$$

Completing the square by adding $(\frac{17}{2})^2$ or $\frac{289}{4}$ to each side,

$$x^2-17x+(\tfrac{17}{2})^2=-30+\tfrac{289}{4}=\tfrac{169}{4},$$
$$\text{or } (x-\tfrac{17}{2})^2=\tfrac{169}{4} \text{ or } (\pm\tfrac{13}{2})^2,$$
$$\therefore\ x-\tfrac{17}{2}=\tfrac{13}{2},$$
$$\therefore\ x=\tfrac{13}{2}+\tfrac{17}{2}=\tfrac{30}{2}=15,$$

that is, A alone can do the work in 15 hours;
and B „ „ „ in 15 hours−5 hours, or 10 hours.

13. Let x=the rate in miles per hour; then

$\frac{40}{x}$=no. hours on the journey

$\frac{40}{x+2}$=no. hours on the journey, travelling 2 miles an hour faster.

But, by the question, the latter is 1 hour less than the former; hence we derive the following equation:

$$\frac{40}{x}-\frac{40}{x+2}=1$$

Multiplying, first by x, and then by $x+2$, we get

$$40\ (x+2)-40x=x\ (x+2),$$
$$\therefore\ 40x+80-40x=x^2+2x,$$
$$\therefore\ x^2+2x=80,$$

Completing the square,

$$x^2+2x+1=81,$$
$$\therefore\ x+1=9,$$
$$\therefore\ x=9-1=8,$$

that is, the rate per hour=8 miles.

44. QUADRATIC EQUATIONS WITH TWO UNKNOWN QUANTITIES.

2.
$$x^2+xy=30\ .\ .\ .\ (1.)$$
$$2x+y=11\ .\ .\ .\ (2.)$$

From eq. (2.), we obtain

$$y=11-2x\ .\ .\ .\ (3.)$$

Substituting this value of y in eq. (1.), we get

$$x^2+x\ (11-2x)=30,$$
$$\therefore\ x^2+11x-2x^2=30,$$
$$\therefore 11x-x^2=30,$$

Changing all the signs, Art. **8.**,

$$x^2-11x=-30$$

Completing the square by adding $(\frac{11}{2})^2$ or $\frac{121}{4}$,

$$x^2-11x+(\tfrac{11}{2})^2=-30+\tfrac{121}{4}=\tfrac{1}{4},$$
$$\therefore\ (x-\tfrac{11}{2})^2=\tfrac{1}{4} \text{ or } (\pm\tfrac{1}{2})^2,$$
$$\therefore\ x-\tfrac{11}{2}=\tfrac{1}{2} \text{ or} -\tfrac{1}{2},$$
$$\therefore\ x=6 \text{ or } 5.$$

Substituting each of these values of x in eq. (3.), we obtain two corresponding values of y:

For $x=6$; $y=11-2x=11-2\times6=-1$.

For $x=5$; $y=11-2x=11-2\times5=1$.

Or thus.

Multiplying eq. (2.) by x, we obtain

$$2x^2+xy=11x$$

Subtracting eq. (1.) from this equation, we get

$$x^2=11x-30,$$
$$\therefore\ x^2-11x=-30,$$

which is the same as the quadratic before found.

3. $$2x^2-xy=12 \quad . \; . \; . \quad (1.)$$
$$x-y=1 \quad . \; . \; . \quad (2.)$$

From eq. (2.), we get

$$y=x-1 \quad . \; . \; . \quad (3.)$$

Substituting this value of y in eq. (1.), we get

$$2x^2-x\,(x-1)=12,$$
$$\therefore\ 2x^2-x^2+x=12;$$
$$\therefore\ x^2+x=12.$$

Completing the square by adding $(\frac{1}{2})^2$, we get

$$x^2+x+\tfrac{1}{4}=12+\tfrac{1}{4}=\tfrac{49}{4},$$
$$\text{or } (x+\tfrac{1}{2})^2=\tfrac{49}{4} \text{ or } (\pm\tfrac{7}{2})^2,$$
$$\therefore\ x+\tfrac{1}{2}=\tfrac{7}{2} \text{ or} -\tfrac{7}{2},$$
$$\therefore\ x=3 \text{ or} -4.$$

Substituting these values of x in eq. (3.), we find

For $x=\ \ 3$; $y=x-1=\ \ 3-1=\ \ 2.$
For $x=-4$; $y=x-1=-4-1=-5.$

Or thus.

Multiplying eq. (2.) by x, we get

$$x^2-xy=x,$$

Subtracting this equation from eq. (1.), we get

$$x^2=12-x,$$
$$\therefore\ x^2+x=12,$$

which is the same as the quadratic equation before found.

4. $$x^2+y^2=74 \quad . \; . \; . \; . \quad (1.)$$
$$x+4y=33 \quad . \; . \; . \; . \quad (2.)$$

From eq. (2.) we find

$$x=33-4y \quad . \; . \; . \; . \quad (3)$$

Substituting this value of x in eq. (1.), we get

$$(33-4y)^2+y^2=74,$$
$$\therefore\ 1089-264y+16y^2+y^2=74,$$
$$\therefore\ 17y^2-264y=-1015$$
$$\therefore\ y^2-\tfrac{264}{17}y=-\tfrac{1015}{17}.$$

Completing the square by adding $\left(\frac{132}{17}\right)^2$ or $\frac{17424}{17\times 17}$,

$$y^2-\frac{264}{17}y+\left(\frac{132}{17}\right)^2=-\frac{1015}{17}+\frac{17424}{17\times 17}=\frac{169}{17\times 17},$$

$$\therefore \left(y-\frac{132}{17}\right)^2=\frac{169}{17\times 17}\text{ or }\left(\pm\frac{13}{17}\right)^2,$$

$$\therefore\ y-\frac{132}{17}=+\frac{13}{17}\text{ or }-\frac{13}{17}.$$

Hence we have for the two values of y,

$$y=\tfrac{13}{17}+\tfrac{132}{17}=\tfrac{145}{17}=8\tfrac{9}{17},$$

$$\text{or } y=-\tfrac{13}{17}+\tfrac{132}{17}=\tfrac{119}{17}=7.$$

Substituting these values of y in eq. (3.), we have

for $y=8\tfrac{9}{17}$; $x=33-4\times 8\tfrac{9}{17}=-1\tfrac{2}{17}$

for $y=7$; $x=33-4\times 7=5$.

5. $xy-y=9\ .\ .\ .\ .\ (1.)$

$x+2y=12\ .\ .\ .\ .\ (2.)$

From eq. (2.), we find

$$x=12-2y\ .\ .\ .\ .\ (3.)$$

Substituting this value of x in eq. (1.), we get

$$(12-2y)\,y-y=9,$$

$$12y-2y^2-y=9,$$

$$\therefore\ 11y-2y^2=9.$$

Changing the signs and dividing by 2, we get

$$y^2-\tfrac{11}{2}y=-\tfrac{9}{2}.$$

Completing the square by adding $(\tfrac{11}{4})^2$ or $\tfrac{121}{16}$,

$$y^2-\tfrac{11}{2}y+(\tfrac{11}{4})^2=-\tfrac{9}{2}+\tfrac{121}{16}=\tfrac{49}{16},$$

$$\therefore\ (y-\tfrac{11}{4})^2=\tfrac{49}{16}\text{ or }(\pm\tfrac{7}{4})^2,$$

$$\therefore\ y-\tfrac{11}{4}=\tfrac{7}{4}\text{ or }-\tfrac{7}{4},$$

$$\therefore\ y=4\tfrac{1}{2}\text{ or }1.$$

Substituting these values of y in eq. (3.), we have

for $y=4\tfrac{1}{2}$; $x=12-2\times 4\tfrac{1}{2}=3$

for $y=1$; $x=12-2\times 1\ =10$.

Or, thus.

Multiplying eq. (2.) by y, we get

$$xy + 2y^2 = 12y.$$

Subtracting eq. (1.) from this, we get

$$2y^2 + y = 12y - 9,$$

$$\therefore\ y^2 - \tfrac{11}{2}y = -\tfrac{9}{2},$$

which is the same as the quadratic equation before found.

6. $$x^2 + y^2 + 4x = 36 \ \ldots\ldots \ (1.)$$

$$\frac{x}{2} + 2y = \ 6 \ \ldots\ldots \ (2.)$$

From eq. (2.), we get

$$y = \frac{12-x}{4} \ \ldots\ldots \ (3.)$$

Substituting this value of y in eq. (1.), we have

$$x^2 + \left(\frac{12-x}{4}\right)^2 + 4x = 36,$$

$$\therefore\ x^2 + \frac{144 - 24x + x^2}{16} + 4x = 36,$$

$$\therefore\ 16x^2 + 144 - 24x + x^2 + 64x = 576,$$

$$\therefore\ 17x^2 + 40x = 432,$$

$$\therefore\ x^2 + \tfrac{40}{17}x = \tfrac{432}{17}.$$

Completing the square by adding $\left(\frac{20}{17}\right)^2$ or $\frac{400}{17 \times 17}$,

$$x^2 + \frac{40}{17}x + \left(\frac{20}{17}\right)^2 = \frac{432}{17} + \frac{400}{17 \times 17} = \frac{7744}{17 \times 17},$$

$$\therefore\ \left(x + \frac{20}{17}\right)^2 = \frac{7744}{17 \times 17} \text{ or } \left(\pm\frac{88}{17}\right)^2.$$

Taking the square root of each side, we get

$$x + \tfrac{20}{17} = \tfrac{88}{17} \text{ or} - \tfrac{88}{17},$$

$$\therefore\ x = \tfrac{68}{17} \text{ or} - \tfrac{108}{17}$$

$$= 4 \text{ or} - 6\tfrac{6}{17}.$$

Substituting these values of x in eq. (3.), we find

$$\text{for } x=4;\ y=\frac{12-4}{4}=2.$$

$$\text{for } x=-6\tfrac{6}{17};\ y=\frac{12-6\frac{6}{17}}{4}=4\tfrac{10}{17}.$$

ART. 45.

2.
$$x-y=1 \quad \ldots . \ (1.)$$
$$xy=20 \quad \ldots . \ (2.)$$

Squaring eq. (1.) as a first step for finding the value of $x+y$,

$$x^2-2xy+y^2=1.$$

Multiplying eq. (2.) by 4, we get

$$4xy=80.$$

Adding this equation to the preceding one,

$$x^2+2xy+y^2=81.$$

Taking the square root of each side of this equation, we get

$$x+y=9 \text{ or } -9 \quad \ldots . \ (3.)$$

Adding this equation to eq. (1.), we get

$$2x=10, \text{ or } -8$$
$$\therefore x=5 \text{ or} -4.$$

Subtracting eq. (1.) from eq. (3.), we get

$$2y=8 \text{ or} -10.$$
$$\therefore y=4 \text{ or} -5.$$

Or, thus.

From eq. (1.), we find

$$x=1+y \quad \ldots . \ (3.)$$

Substituting this value of x in eq. (2.)

$$(1+y)\,y=20,$$
$$\therefore y^2+y=20,$$
$$\therefore y^2+y+\tfrac{1}{4}=20+\tfrac{1}{4}=\tfrac{81}{4}$$

$$\therefore\ y+\tfrac{1}{2}=\tfrac{9}{2} \text{ or } -\tfrac{9}{2},$$

$$\therefore\ y=4 \text{ or } -5.$$

Substituting these values of y in eq. (3.), we find

$$\text{for } y=\ \ 4;\ x=1+4=\ \ 5,$$

$$\text{for } y=-5;\ x=1-5=-4.$$

4. $$x+y=9 \ \ldots\ldots \ (1.)$$

$$x^2+y^2=53 \ \ldots\ldots \ (2.)$$

Squaring eq. (1.), as a first step for finding the value of $x-y$,

$$x^2+2xy+y^2=81,$$

Subtracting eq. (2.) from this equation, we get

$$2xy=28,$$

Subtracting this eq. from eq. (2.),

$$x^2-2xy+y^2=25,$$

Taking the square root,

$$x-y=5, \text{ or} -5 \ \ldots\ldots \ (3.)$$

Adding this eq. to eq. (1.),

$$2x=14 \text{ or } 4$$

$$\therefore\ x=7 \text{ or } 2$$

Subtracting eq. (3) from eq. (1.),

$$2y=4 \text{ or } 14$$

$$\therefore\ y=2 \text{ or } 7.$$

Or, thus.

From eq. (1.) we find

$$x=9-y \ \ldots\ldots \ (3.)$$

Substituting this value of x in eq. (2.), we get

$$(9-y)^2+y^2=53,$$

$$\therefore\ 81-18y+y^2+y^2=53,$$

$$\therefore\ 2y^2-18y=-28,$$

$$\therefore\ y^2-9x=-14,$$

$$\therefore\ y^2-9x+(\tfrac{9}{2})^2=-14+\tfrac{81}{4}=\tfrac{25}{4},$$

$$\therefore\ y-\tfrac{9}{2}=\tfrac{5}{2} \text{ or} -\tfrac{5}{2}$$

$$\therefore\ y=7 \text{ or } 2.$$

Substituting these values of y in eq. (3.), we find

$$\text{for } y=7\,;\ x=9-7=2$$
$$\text{for } y=2\,;\ x=9-2=7.$$

5. $$x^2+y^2=17 \ \ldots\ldots \ (1.)$$
$$xy=4 \ \ldots\ldots \ (2.)$$

Adding twice the second equation to the first, we get

$$x^2+2xy+y^2=25,$$
$$\therefore\ (x+y)^2=(\pm 5)^2$$
$$\therefore\ x+y=\pm 5 \ \ldots\ldots \ (3.)$$

Subtracting twice eq. (2.) from eq. (1.), we get

$$x^2-2xy+y^2=9,$$
$$\therefore\ (x-y)^2=(\pm 3)^2,$$
$$\therefore\ x-y=\pm 3 \ \ldots\ldots \ (4.)$$

Now equations (3.) and (4.) really give us four pairs of simultaneous equations, and therefore corresponding values of x and y may be determined from each pair; thus

$$\begin{array}{llll} & \left.\begin{array}{l} x+y=5 \\ \underline{x-y=3} \end{array}\right\} & & \begin{array}{l} x+y=\ \ 5 \\ \underline{x-y=-3} \end{array} \\ \text{adding} & 2x=8,\ \therefore\ x=4. & & 2x=\ \ 2,\ \therefore\ x=1. \\ \text{subt.} & 2y=2,\ \therefore\ y=1. & & 2y=\ \ 8,\ \therefore\ y=4. \end{array}$$

These are the two pairs of roots given in the text.

Similarly by combining $x+y=-5$, with each of the two equations $x-y=\pm 3$, we get two other pairs of roots:

$$\begin{array}{ll} \begin{array}{l} x+y=-5 \\ \underline{x-y=\ \ 3} \end{array} & \begin{array}{l} x+y=-5 \\ \underline{x-y=-3} \end{array} \\ \text{adding } 2x=-2,\ \therefore\ x=-1 & \text{adding } 2x=-8,\ \therefore\ x=-4 \\ \text{subt. } 2y=-8,\ \therefore\ y=-4 & \text{subt. } 2y=-2,\ \therefore\ y=-1 \end{array}$$

Hence it appears that the latter two pairs of roots are the former with their signs changed.

7. $$x^2-y^2=12 \ \ldots\ldots \ (1.)$$
$$x-y=\ 2 \ \ldots\ldots \ (2.)$$

By Art. **34.** eq. (1.) may be written

$$(x+y)(x-y)=12,$$

Substituting the value of $x-y$ given in eq. (2.),

$$(x+y)\times 2=12,$$

$$\therefore\ x+y=6\ \ldots.\ (3.)$$

Adding this equation to eq. (2.), we find

$$2x=8,$$

$$\therefore\ x=4.$$

Subtracting eq. (2.) from eq. (3.), we find

$$2y=4,$$

$$\therefore\ y=2.$$

10. $$x-y=1\ \ldots.\ (1.)$$

$$x^3-y^3=7\ \ldots.\ (2.)$$

Taking the cube of the first equation, we get

$$x^3-3x^2y+3xy^2-y^3=1,$$

Subtracting this equation from eq. (2.), we get

$$3x^2y-3xy^2=6,$$

$$\therefore\ x^2y-xy^2=2,$$

$$\therefore\ xy(x-y)=2,$$

Substituting the value of $x-y$ given in eq. (1.), we get

$$xy=2,$$

Adding four times this equation to the square of eq. (1.),

$$x^2+2xy+y^2=9,$$

$$\therefore\ x+y=3 \text{ or} -3\ \ldots.\ (3.)$$

Adding eq. (1.) to this equation, we get

$$2x=4 \text{ or} -2$$

$$\therefore\ x=2 \text{ or} -1$$

Subtracting eq. (1.) from eq. (3.), we get

$$2y=2 \text{ or } -4$$

$$\therefore\ y=1 \text{ or } -2.$$

11. $$x^2+y^2+xy=19\ \ldots.\ (1.)$$

$$x+y=5\ \ldots.\ (2.)$$

Squaring eq. (2.), we get

$$x^2+2xy+y^2=25.$$

Subtracting eq. (1.) from this equation, we get

$$xy=6.$$

Subtracting 3 times this equation from eq. (1.), we get

$$x^2-2xy+y^2=19-18=1,$$
$$\therefore\ (x-y)^2=1,$$
$$\therefore\ x-y=1 \text{ or } -1 \quad \ldots\ldots \quad (3.)$$

Adding this equation to eq. (2.),

$$2x=6 \text{ or } 4,$$
$$\therefore\ x=3 \text{ or } 2.$$

Subtracting eq. (3.) from eq. (2.),

$$2y=4 \text{ or } 6,$$
$$\therefore\ y=2 \text{ or } 3.$$

12. $$x^2+xy=\ 6 \quad \ldots\ldots \quad (1.)$$
$$y^2+xy=30 \quad \ldots\ldots \quad (2.)$$

Adding these equations together, we get

$$x^2+2xy+y^2=36,$$
$$\therefore\ (x+y)^2=(\pm 6)^2,$$
$$\therefore\ x+y=6 \text{ or } -6 \quad \ldots\ldots \quad (3.)$$

From eq. (1.) we get by simply taking out the factor,

$$(x+y)x=6.$$

Substituting the two values of $x+y$, given in eq. (3.), we have

First, $6x=6,\ \therefore\ x=1.$

Second, $-6x=6,\ \therefore\ x=-1.$

In like manner, from eq. (2.), we get

$$(x+y)y=30.$$

Substituting the two values of $x+y$, given in eq. (3.), we have

First, $6y=30,\ \therefore\ y=5.$

Second, $-6y=30,\ \therefore\ y=-5.$

13. $$x^3+y^3=9 \quad .\ .\ .\ .\ (1.)$$
$$x^2y+xy^2=6 \quad .\ .\ .\ .\ (2.)$$

Taking 3 times eq. (2.), and adding the result to eq. (1.),

$$x^3+3x^2y+3xy^2+y^3=9+18=27,$$
$$\therefore\ (x+y)^3=3^3,$$
$$\therefore\ x+y=3 \quad .\ .\ .\ .\ (3.)$$

From eq. (2.) we find

$$xy(x+y)=6.$$

Substituting the value of $x+y$, given in eq. (3.), we get

$$3xy=6,$$
$$\therefore\ 4xy=8.$$

Subtracting this equation from the square of eq. (3.), we get

$$x^2-2xy+y^2=9-8=1,$$
$$\therefore\ (x-y)^2=1,$$
$$\therefore\ x-y=1 \text{ or } -1 \quad .\ .\ .\ .\ (4.)$$

Adding this eq. to eq. (3.),

$$2x=4 \text{ or } 2,$$
$$\therefore\ x=2 \text{ or } 1.$$

Subt. eq. (4.) from eq. (3.),

$$2y=2 \text{ or } 4,$$
$$\therefore\ y=1 \text{ or } 2.$$

15. $$x^2+y^2+x-y=6 \quad .\ .\ .\ .\ (1.)$$
$$xy=2 \quad .\ .\ .\ .\ (2.)$$

Subtracting twice eq. (2.) from eq. (1.), we get

$$x^2-2xy+y^2+x-y=6-4=2,$$
$$\therefore\ (x-y)^2+x-y=2.$$

This equation may be solved as a quadratic for the value of $x-y$; then adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to complete the square, we have

$$(x-y)^2+(x-y)+\tfrac{1}{4}=2+\tfrac{1}{4}=\tfrac{9}{4},$$
$$\text{or } (\overline{x-y}+\tfrac{1}{2})^2=(\pm\tfrac{3}{2})^2,$$
$$\therefore\ x-y+\tfrac{1}{2}=\tfrac{3}{2} \text{ or } -\tfrac{3}{2},$$
$$\therefore\ x-y=1 \text{ or } -2.$$

This result contains two equations, viz., $x-y=1$, and $x-y=-2$; and, therefore, each of these equations combined with eq. (2.) will give two values for x and y.

First, taking the equation, $x-y=1$ . . . (3.)
Squaring, $x^2-2xy+y^2=1$,
Adding 4 times eq. (2.), $x^2+2xy+y^2=9$,
Taking the sq. root, $x+y=3$ or -3 . . . (4.)
Adding eq. (3.) and eq. (4.), $2x=4$ or -2,
$\therefore\ x=2$ or -1.
Subt. eq. (3.) from eq. (4.), $2y=2$ or -4,
$\therefore\ y=1$ or -2.

These are the two pairs of roots given in the text.

Second, taking the equation, $x-y=-2$, . . . (5.)
Squaring, $x^2-2xy+y^2=4$,
Adding 4 times eq. (2.) $x^2+2xy+y^2=12$,
Taking the sq. root, $x+y=\pm\sqrt{12}$ . . . (6.)
Adding eq. (5.) and eq. (6.), $2x=\pm\sqrt{12}-2$,
$\therefore\ x=\frac{1}{2}(\pm\sqrt{12}-2)$.
Subt. eq. (5.) from eq. (6.) $2y=\pm\sqrt{12}+2$,
$\therefore\ y=\frac{1}{2}(\pm\sqrt{12}+2)$.

These expressions, for x and y, give the values of two other pairs of roots, which may be calculated approximately in decimals.

16. $x-y=2$ . . . . (1.)
$x^4+y^4=272$ . . . . (2.)

Raising eq. (1.) to the 4th power, Art. **35**,

$$x^4-4x^3y+6x^2y^2-4xy^3+y^4=2^4=16.$$

Subtracting this equation from eq. (2.), we find

$$4x^3y-6x^2y^2+4xy^3=272-16=256,$$
$$\therefore\ 2x^3y-3x^2y^2+2xy^3=128.$$

Subtracting x^2y^2 from each side of this equation,

$$2x^3y-4x^2y^2+2xy^3=128-x^2y^2,$$
$$\therefore\ 2xy(x^2-2xy+y^2)=128-x^2y^2.$$

But the square of eq. (1.) gives $x^2-2xy+y^2=4$, therefore substituting this value in the last equation, we get

$$8xy=128-x^2y^2,$$
$$\therefore\ x^2y^2+8xy=128.$$

Solving this equation as a quadratic for the value of xy, we complete the square by adding 4^2 or 16,

$$x^2y^2+8xy+16=128+16=144,$$
$$\therefore\ (xy+4)^2=144 \text{ or } (\pm 12)^2,$$
$$\therefore\ xy+4=\ 12 \text{ or} -12,$$
$$\therefore\ xy=\ 8 \text{ or} -16.$$

This result contains two equations, viz., $xy=8$, and $xy=-16$; and therefore each of these equations, combined with eq. (1.), will give two values for x and y.

Taking the equation, $xy=8$ . . . (3.)

Adding the square of eq. (1.) to 4 times eq. (3.), we get

$$x^2+2xy+y^2=36,$$

Taking the sq. root, $x+y=6$ or -6 . . . (4.)

Adding eq. (1.) to eq. (4.), $2x=8$ or -4,

$\therefore\ x=4$ or -2.

Subt. eq. (1.) from eq. (4.), $2y=4$ or -8,

$\therefore\ y=2$ or -4.

These are the roots determined in the text by another method.

Similarly two other pairs of roots may be obtained from the equation $xy=-16$.

17. $x+y=\ 4$ . . . (1.)

$x^4+y^4=82$ . . . (2.)

Putting $x=u+v$, and $y=u-v$; then eq. (1.) becomes

$$\overline{u+v}+\overline{u-v}=4,$$
$$\therefore\ 2u=4,$$
$$\therefore\ u=2.$$

And eq. (2.) becomes

$$(u+v)^4+(u-v)^4=82,$$

Developing the binomials, Art. **35.**, and adding the results, we obtain

$$2u^4+12u^2v^2+2v^4=82,$$
$$\therefore\ u^4+6u^2v^2+v^4=41.$$

Substituting 2 for u (as found above), we have

$$16+24v^2+v^4=41,$$
$$\therefore\ v^4+24v^2=25.$$

Solving this equation as a quadratic, Ex. 25, Art. **42.**,

$$v^4+24v^2+144=25+144=169,$$
$$\therefore\ (v^2+12)^2=169 \text{ or } 13^2,$$
$$\therefore\ v^2+12=13$$
$$\therefore\ v^2=1$$
$$\therefore v=\pm 1.$$

The imaginary roots being neglected,

Then $$x=u+v=2\pm 1=3 \text{ or } 1$$
$$y=u-v=2+1=1 \text{ or } 3.$$

Or thus.

$$x+y=\ 4\ .\quad .\quad .\quad (1.)$$
$$x^4+y^4=82\ .\quad .\quad .\quad (2.)$$

Raising eq. (1.) to the 4th power, and subtracting eq. (2.) from the result, we obtain

$$4x^3y+6x^2y^2+4xy^3=174,$$
$$\therefore\ 2x^3y+3x^2y^2+2xy^3=\ 87.$$

Adding x^2y^2 to each side, and reducing,

$$2xy\ (x^2+2xy+y^2)=87+x^2y^2,$$
$$\therefore\ 2xy(x+y)^2=87+x^2y^2.$$

Substituting the value of $x+y$ given in eq. (1.)

$$32xy=87+x^2y^2,$$
$$\therefore\ x^2y^2-32xy=-87.$$

Solving this equation as a quadratic for xy,

$$x^2y^2-32xy+16^2=-87+16^2=169,$$
$$\therefore\ (xy-16)^2=169 \text{ or } (\pm 13)^2,$$
$$\therefore\ xy-16=13 \text{ or} -13,$$
$$\therefore\ xy=29 \text{ or } 3.$$

This result contains two equations, viz., $xy=29$, and $xy=3$; and therefore each of these equations, combined with eq. (1.), will give two values for x and y.

Taking, $xy=3$ . . . (3.)

Squaring eq. (1.), and subtracting 4 times eq. (3.) from the result,

$$x^2-2xy+y^2=4.$$

Taking the sq. root, $\therefore x-y=2 \text{ or} -2$ . . . (4.)

Adding eq. (4.) and (1.), $2x=6 \text{ or } 2$

$\therefore x=3 \text{ or } 1$

Subt. eq. (4.) from (1.), $2y=2 \text{ or } 6$

$\therefore y=1 \text{ or } 3$

Art. 46.

2. $2x^2-3xy=8$ . . . (1.)

$4y^2+3x^2=64$ . . . (2.)

Putting zx for y, the two equations become

$$2x^2-3zx^2=8; \quad \therefore x^2=\frac{8}{2-3z} \quad . . . . (3.)$$

$$4z^2x^2+3x^2=64; \therefore x^2=\frac{64}{4z^2+3}.$$

Therefore by equating the values for x^2, we get

$$\frac{8}{2-3z}=\frac{64}{4z^2+3},$$

$$\therefore \frac{1}{2-3z}=\frac{8}{4z^2+3},$$

$$\therefore 4z^2+3=16-24z,$$

$$\therefore z^2+6z=\tfrac{13}{4}.$$

Completing the square by adding 3^2 or 9 to each side,

$$z^2+6z+9=\tfrac{13}{4}+9=\tfrac{49}{4},$$

$$\therefore (z+3)^2=\tfrac{49}{4} \text{ or } (\pm\tfrac{7}{2})^2,$$

$$\therefore z+3=\tfrac{7}{2} \text{ or} -\tfrac{7}{2},$$

$$\therefore z=\tfrac{1}{2} \text{ or } -\tfrac{13}{2}.$$

Substituting $\frac{1}{2}$ for z in eq. (3), we get

$$x^2=\frac{8}{2-3z}=\frac{8}{2-3\times\frac{1}{2}}=16,$$

$$\therefore\ x=4 \text{ or} -4.$$

$$\text{and } y=zx=\tfrac{1}{2}\times\pm4=\pm2=2 \text{ or} -2.$$

These are the roots given in the text; but other roots might be found from the equation $z=-\frac{13}{2}$.

3. $$x^2+2y^2=57 \quad . \quad . \quad . \quad . \quad (1.)$$

$$3x^2+xy=95 \quad . \quad . \quad . \quad . \quad (2.)$$

Putting zx for y, the two equations become

$$x^2+2z^2x^2=57;\ \therefore\ x^2=\frac{57}{1+2z^2};$$

$$3x^2+zx^2=95;\ \therefore\ x^2=\frac{95}{3+x} \quad . \quad . \quad . \quad . \quad (3.)$$

Therefore by equating the values for x^2, we get

$$\frac{95}{3+z}=\frac{57}{1+2z^2},$$

$$\therefore\ 95+190z^2=171+57z,$$

$$\therefore\ 190z^2-57z=76,$$

$$\therefore\ z^2-\tfrac{57}{190}\,z=\tfrac{76}{190}=\tfrac{38}{95}.$$

Completing the square by adding $\left(\frac{57}{380}\right)^2$, we have

$$z^2-\frac{57}{190}z+\left(\frac{57}{380}\right)^2=\frac{38}{95}+\left(\frac{57}{380}\right)^2=\frac{61009}{380^2},$$

$$\therefore\ \left(z-\frac{57}{380}\right)^2=\frac{61009}{380^2} \text{ or } \left(\pm\frac{247}{380}\right)^2,$$

$$\therefore\ z-\tfrac{57}{380}=\tfrac{247}{380} \text{ or } -\tfrac{247}{380}$$

$$\therefore\ z=\tfrac{304}{380} \text{ or } -\tfrac{190}{380},$$

$$=\tfrac{4}{5} \text{ or } -\tfrac{1}{2}.$$

Substituting $\frac{4}{5}$ for z in eq. (3.), we get

$$x^2=\frac{95}{3+z}=\frac{95}{3+\frac{4}{5}}=\frac{475}{19}=25,$$

$$\therefore\ x=\pm\sqrt{25}=5 \text{ or } -5;$$

$$\text{and } y=zx=\tfrac{4}{5}\times\pm5=4 \text{ or } -4.$$

These are the two roots of x and y given in the text; but other roots may be found from the equation $z=-\frac{1}{2}$. Thus we find $x=\pm\sqrt{38}$, and $y=\mp\frac{1}{2}\sqrt{38}$.

4. $$2x^2-y^2=14 \quad . \quad . \quad . \quad . \quad (1.)$$
$$2y^2-x^2=-1 \quad . \quad . \quad . \quad . \quad (2.)$$

Putting zx for y, the two equations become

$$2x^2-z^2x^2=14;\ \therefore\ x^2=\frac{14}{2-z^2} \quad . \quad . \quad . \quad . \quad (3.)$$
$$2z^2x^2-x^2=-1;\ \therefore\ x^2=\frac{-1}{2z^2-1}.$$

Therefore by equating the values for x^2, we get

$$\frac{14}{2-z^2}=-\frac{1}{2z^2-1},$$
$$\therefore\ 28z^2-14=z^2-2,$$
$$\therefore\ 27z^2=12,$$
$$\therefore\ z^2=\tfrac{12}{27}=\tfrac{4}{9},$$
$$\therefore\ z=\tfrac{2}{3}\text{ or }-\tfrac{2}{3}.$$

Substituting $\frac{2}{3}$ for z in eq. (3.), we have

$$x^2=\frac{14}{2-z^2}=\frac{14}{2-\frac{4}{9}}=\frac{126}{18-4}=\frac{126}{14}=9;$$
$$\therefore\ x=\pm\sqrt{9}=3\text{ or}-3;$$
$$\text{and } y=zx=\tfrac{2}{3}\times\pm3=2\text{ or}-2.$$

These are the values of x and y given in the text.

In like manner, substituting $-\frac{2}{3}$ for z in eq. (3.), we find $x=3$ or -3, as before; and $y=-2$ or $+2$.

Or, thus.

Adding twice eq. (2.) to eq. (1.), we get

$$3y^2=12,$$
$$\therefore\ y^2=4,$$
$$\therefore\ y=2\text{ or}-2\text{, as before;}$$

and substituting these values in eq. (1.), we get

$$2x^2-4=14,$$
$$\therefore\ 2x^2=18,$$
$$\therefore\ x^2=9,$$
$$\therefore\ x=3 \text{ or} -3, \text{ as before.}$$

47. PROBLEMS IN SIMULTANEOUS EQUATIONS OF TWO UNKNOWN QUANTITIES.

2. Let x=the one number,
and y=the other number ; then
xy=their product, and
x^2+y^2=the sum of their squares.

Hence, by the question, we have the two following equations:

$$xy=6 \ .\ .\ .\ .\ (1.)$$
$$x^2+y^2=13 \ .\ .\ .\ .\ (2.)$$

Adding twice eq. (1.) to eq. (2.), we get

$$x^2+2xy+y^2=13+12=25,$$
$$\therefore\ (x+y)^2=25 \text{ or } (\pm 5)^2,$$
$$\therefore\ x+y=5 \text{ or} -5 \ .\ .\ .\ .\ (3.)$$

Subtracting twice eq. (1.) from eq. (2.), we get

$$x^2-2xy+y^2=13-12=1,$$
$$\therefore\ (x-y)^2=1 \text{ or } (\pm 1)^2,$$
$$\therefore\ x-y=1 \text{ or} -1 \ .\ .\ .\ .\ (4.)$$

Adding eq. (4.) to eq. (3.), we get

$$2x=6 \text{ or} -6$$
$$\therefore\ x=3 \text{ or} -3.$$

Subtracting eq. (4.) from eq. (3.), we get

$$2y=4 \text{ or} -4$$
$$\therefore\ y=2 \text{ or} -2.$$

Hence the required numbers are 2 and 3; but they may also be -2 and -3.

3. Let x=the ten's digit,
and y=the unit's digit; then
$10x+y$=the number.

Then from the conditions of the problem, we find the two following equations:

$$\frac{10x+y}{x+y}=4 \quad \ldots . \quad (1.)$$

$$xy=\tfrac{1}{2}(10x+y) \quad \ldots . \quad (2.)$$

Reducing eq. (1.), we get

$$10x+y=4x+4y,$$

$$\therefore\ 6x=3y,$$

$$\therefore\ y=\frac{6x}{3}=2x \quad \ldots . \quad (3.)$$

Substituting this value of y in eq. (2.), we get

$$x\times 2x=\tfrac{1}{2}(10x+2x)=6x$$

$$\therefore\ x=\frac{6x}{2x}=3.$$

Substituting this value of x in eq. (3.), we get

$$y=2x=2\times 3=6.$$

Hence the number = $10x+y=10\times 3+6=36$.

4. Let x=the numerator of the fraction,
and y=the denominator „ „ then
$\frac{x}{y}$=the fraction.

From the data of the question, we find the two following equations:

$$xy=12 \quad \ldots . \quad (1.)$$

$$\frac{x+5}{y}=2 \quad \ldots . \quad (2.)$$

From eq. (2.), we find

$$x+5=2y,$$

$$\therefore\ x=2y-5 \quad \ldots . \quad (3.)$$

Substituting this value of x in eq. (1.),

$$(2y-5)\,y=12,$$
$$\therefore\ 2y^2-5y=12$$
$$\therefore\ y^2-\tfrac{5}{2}y=6.$$

Completing the square by adding $(\frac{5}{4})^2$ or $\frac{25}{16}$ to each side,

$$y^2-\tfrac{5}{2}\,y+(\tfrac{5}{4})^2=6+\tfrac{25}{16}=\tfrac{121}{16},$$
$$\therefore\ (y-\tfrac{5}{4})^2=\tfrac{121}{16}\text{ or }(\pm\tfrac{11}{4})^2,$$
$$\therefore\ y-\tfrac{5}{4}=\tfrac{11}{4}\text{ or}-\tfrac{11}{4},$$
$$\therefore\ y=4\text{ or}-\tfrac{3}{2}.$$

Substituting 4 for y in eq. (3.), we get

$$x=2y-5=2\times 4-5=3.$$

Hence the fraction$=\dfrac{x}{y}=\frac{3}{4}$.

5. Let x=the distance, in miles,
and y=the rate per hour, in miles.

$$\therefore\ \text{No. hours on the journey}=\frac{x}{y}.$$

No. hours on the journey, going 1 mile quicker $=\dfrac{x}{y+1}$;

but, from the question, this is equal to $\frac{3}{4}$ of $\dfrac{x}{y}$;

$$\therefore\ \frac{x}{y+1}=\frac{3}{4}\times\frac{x}{y}.$$

Reducing this equation,

$$\frac{1}{y+1}=\frac{3}{4}\times\frac{1}{y}.$$
$$\therefore\ 4y=3\,(y+1)=3y+3,$$
$$\therefore\ y=3,\ \text{No. miles per hour.}$$

Again, to find another equation, we have

No. hours on the journey, going 1 mile slower$=\dfrac{x}{y-1}$;

but, in this case, he is 5 hours longer on the journey;

$$\therefore\ \frac{x}{y-1}=\frac{x}{y}+5.$$

Substituting the value of y (before found), we get

$$\frac{x}{3-1}=\frac{x}{3}+5,$$

$$\therefore\ \frac{x}{2}=\frac{x}{3}+5,$$

Multi. by 6, $$3x=2x+30,$$

$$\therefore\ x=30, \text{ the distance in miles.}$$

6. $$\left.\begin{aligned}\frac{3}{x}-\frac{3}{y}&=4\\ 8x+10y&=19.\end{aligned}\right\}$$

Reducing the first equation, we have

$$3y-3x=4xy\ \ldots\ldots\ (1.)$$

From the second equation, we find

$$x=\frac{19-10y}{8}\ \ldots\ldots\ (2.)$$

Substituting this in eq. (1.), we have

$$3y-\frac{3(19-10y)}{8}=4y\times\frac{19-10y}{8},$$

Multi. by 8, $$24y-57+30y=76y-40y^2,$$

$$\therefore\ 40y^2-22y=57,$$

$$\therefore\ y^2-\tfrac{11}{20}y=\tfrac{57}{40}.$$

Completing the square by adding $(\frac{11}{40})^2$,

$$y^2-\tfrac{11}{20}y+(\tfrac{11}{40})^2=\tfrac{57}{40}+(\tfrac{11}{40})^2=\tfrac{2401}{1600},$$

$$\therefore\ y-\tfrac{11}{40}=\sqrt{\tfrac{2401}{1600}}=\tfrac{49}{40},$$

$$\therefore\ y=\tfrac{49}{40}+\tfrac{11}{40}=\tfrac{60}{40}=\tfrac{3}{2}s.=18d.$$

Substituting $\frac{3}{2}$ for y in eq. (2.), we find

$$x=\frac{19-10\times\frac{3}{2}}{8}=\frac{4}{8}=\frac{1}{2}s.=6d.$$

7. Let $x=$ the price of each sheep, in shillings
and $y=$ " " lamb, "

Price of 3 lambs and 1 sheep, in shillings $=3y+x$;
But this is equal to 36 shillings,

$$\therefore\ 3y+x=36\ \ldots\ldots\ (1.)$$

Again, to determine another equation, we have

No. sheep to cost 48*s*. $=\frac{48}{x}$;

„ lambs „ „ $=\frac{48}{y}$;

but, by the question, the latter is 2 more than the former,

$$\therefore\ \frac{48}{y}-\frac{48}{x}=2.$$

Reducing this equation,

$$\frac{24}{y}-\frac{24}{x}=1,$$

Multi. by xy, $24x-24y=xy$ . . . . (2.)

From eq. (1.), we find,

$$x=36-3y \ . \ . \ . \ . \ (3.$$

Substituting this in eq. (2.), we get

$$24(36-3y)-24y=(36-3y)y,$$
$$\therefore\ 864-72y-24y=36y-3y^2,$$
$$\therefore\ 3y^2-132y=-864,$$
$$\therefore\ y^2-44y=-288.$$

Completing the square by adding 22^2 or 484 to each side,

$$y^2-44y+22^2=-288+484=196,$$
$$\therefore\ (y-22)^2=196 \text{ or } (\pm 14)^2,$$
$$\therefore\ y-22=14 \text{ or} -14,$$
$$\therefore\ y=36 \text{ or } 8.$$

Rejecting 36 as being inconsistent with the conditions of the problem, we have

$y=8s.$, the price of each lamb;

and substituting 8 for y in eq. (3.), we get

$x=36-3y=36-3\times 8=12s.$, the price of each sheep.

48. LITERAL EQUATIONS.

4. Let x=the greater,
and y=the less; then
$x+y$=the sum
$x-y$=the difference

$$\therefore \left.\begin{array}{l} x+y=s \\ x-y=d \end{array}\right\}$$

Adding, $$2x=s+d$$

$$\therefore\ x=\frac{s+d}{2}.$$

5. Let x=the number;

$$\therefore\ x+a=bx,$$

By transposition, $x-bx=-a,$

Changing the signs, $bx-x=\ \ a,$

$$(b-1)\,x=\ \ a,$$

$$\therefore\ x=\frac{a}{b-1}.$$

6. Let x=the No. of the 1st class,
and y= „ „ 2nd „ ; then
$x+y$=the whole number of persons,
cx=the No. shillings given to the 1st
dy= „ „ „ 2nd

By the question the total number of persons was b;

$$\therefore\ x+y=b\ \ .\ .\ .\ (1.)$$

The total No. shillings distributed$=cx+dx$;
but by the question, this is equal to a shillings,

$$\therefore\ cx+dy=a\ \ .\ .\ .\ (2.)$$

Multiplying eq. (1.) by d and subtracting the result fron eq. (2.),

$$cx-dx=a-bd,$$

$$\therefore\ (c-d)x=a-bd,$$

$$\therefore\qquad x=\frac{a-bd}{c-d}.$$

Multiplying eq. (1.) by c and subtracting eq. (2.) from the result,

$$cy-dy=bc-a,$$
$$\therefore\ (c-d)\,y=bc-a,$$
$$\therefore\qquad y=\frac{bc-a}{c-d}.$$

By changing the signs of the quantities in the numerator, and also in denominator, these results may be also written,

$$x=\frac{bd-a}{d-c},\ \text{and}\ y=\frac{a-bc}{d-c}.$$

9. Let x=No. hours B takes to overtake A; then

$x+c$=No. hours A is travelling.

∴ Distance gone by B$=dx$,

" " A$=b\,(x+c)$;

But these distances must be equal to each other,

$$\therefore\ dx=b\,(x+c),$$
$$\therefore\ dx=bx+bc,$$
$$\therefore\ dx-bx=bc,$$
$$\therefore\ (d-b)x=bc,$$
$$\therefore\qquad x=\frac{bc}{d-b}.$$

10. Let x=the distance, in miles.

Here we shall find two independent expressions for the time.

No. hours the 1st will take to travel the distance$=\frac{x}{m}$;

" 2nd " " " $=\frac{x}{m_1}$;

But the former is h hours greater than the latter;

$$\therefore\ \frac{x}{m}-\frac{x}{m_1}=h,$$
$$\therefore\ x\left(\frac{1}{m}-\frac{1}{m_1}\right)=h,$$
$$\therefore\ x=\frac{h}{\frac{1}{m}-\frac{1}{m_1}}=\frac{m_1mh}{m_1-m}.$$

11. Let x=the rate of the tide, in miles per hour;
and y=the rate of the boat, in miles per hour, supposing it on still water; then

$$\left.\begin{array}{r} x+y=a, \\ \text{and } y-x=b \end{array}\right\}$$

Therefore by subtraction, we find

$$2x=a-b,$$

$$\therefore\ x=\frac{a-b}{2}.$$

50. PROBLEMS REQUIRING THE USE OF PROPORTION.

2. Let x=the 1st part; then
$20-x$=the 2nd part.

And by the given ratio, we have

$$x : 20-x : : 2 : 3,$$

Taking the product of the extremes and means, we have

$$3\times x= (20-x)\times 2,$$

$$\therefore\ 3x=40-2x,$$

$$\therefore\ 5x=40,\ \therefore\ x=\tfrac{40}{5}=8, \text{ the 1st part;}$$

and the 2nd part$=20-x=20-8=12$.

3. Let x=the greater number,
and y=the less number,

Then from the data of the question, we have

$$xy=24\ (x-y)\ .\ .\ .\ (1.)$$

But we have also, from the given ratio,

$$x : y : : 4 : 3,$$

$$\therefore\ 3x=4y,$$

$$\text{and } x=\tfrac{4}{3}y\ .\ .\ .\ (2.)$$

Substituting this value of x in eq. (1.), we have

$$\tfrac{4}{3}y^2=24\ (\tfrac{4}{3}y-y)=32y-24y=8y$$

$$\therefore\ \tfrac{4}{3}y=8,$$

$$\therefore\ y=\frac{8\times 3}{4}=6, \text{ the less.}$$

Substituting 6 for y in eq. (2.), we get

$$x=\tfrac{4}{3}y=\tfrac{4}{3}\times 6=8, \text{ the greater.}$$

4. Let $x=$the No. hours they take to meet; then
$x+4=$the No. hours A takes to complete the journey,
$x+9=$ „ D „ „ „ .

Let A and D be the two places and M the point of meeting; then

$$\left.\begin{array}{l}\text{No. hours A takes to travel AM}=x\\ \quad\text{,, D ,, ,, AM}=9\end{array}\right\}$$
$$\left.\begin{array}{l}\quad\text{,, A ,, ,, DM}=4\\ \quad\text{,, D ,, ,, DM}=x\end{array}\right\}$$

Now their rates of travelling will be inversely as the times which they take in going the same distance; hence we have

$$\text{Rate A : Rate D} :: \frac{1}{x} : \frac{1}{9},$$

$$\text{and Rate A : Rate D} :: \frac{1}{4} : \frac{1}{x};$$

Therefore, we have, by this equality of ratio,

$$\frac{1}{x} : \frac{1}{9} :: \frac{1}{4} : \frac{1}{x},$$

$$\therefore \frac{1}{x^2}=\frac{1}{9}\times\frac{1}{4}=\frac{1}{36},$$

$$\therefore \frac{1}{x}=\frac{1}{6}, \therefore x=6.$$

Then, A's time$=x+4=6+4=10$ hours.
D's time$=x+9=6+9=15$ hours.

Or thus.

Let $a=$A's rate in miles per hour,
and $b=$D's „ „ „ ; then

$$\left.\begin{array}{l}\text{The distance AM}=a\times x\\ \quad\text{,, ,, }=b\times 9\end{array}\right\}\therefore ax=9b.$$

$$\left.\begin{array}{l}\text{The distance DM}=a\times 4\\ \quad\text{,, ,, }=b\times x\end{array}\right\}\therefore bx=4a.$$

Multiplying these equations together, we get

$$abx^2=36ab.$$

Dividing each side of the equation by ab,

$$x^2=36$$

$$\therefore\ x=6, \text{ as before.}$$

5. Let x=the less part; then

$x+10$=the greater part.

But by the question, we have

$$x+\overline{x+10}=20,$$

$$\therefore\ 2x=20-10=10,$$

$$\therefore\ x=5, \text{ the less;}$$

and the greater$=x+10=5+10=15$;

then $5:15::1:3$.

51. ARITHMETICAL SERIES.

Examples. (To find any term in an arithmetical series.)

2. Here the 1st term=1, and the common difference=4;
$\therefore$ the 10th term$=1+(10-1)4=1+9\times4=37$.

3. Here the 1st term=7, and the common difference=1;
$\therefore$ the 41st term$=7+(41-1)1=7+40=47$.

4. Here the 1st term=3, and the common difference=5;
$\therefore$ the 11th term$=3+(11-1)5=3+50=53$;
and the term preceding this$=53-5=48$.

Examples. (To find the sum of an arithmetical series.)

2. Here the 1st term=1, the common difference=3, and the 18th term$=1+(18-1)3=1+17\times3=52$; hence the series may be written down,

$$s=1+4+7+\ \ldots\ldots\ +46+49+52,$$

writing the terms in an inverse order,

$$s=52+49+46+\ \ldots\ldots\ +7+4+1.$$

Adding the corresponding terms together,

$$2s=53+53+53+\ \ldots\ +53+53+53.$$

But there are 18 terms in this series,

$$\therefore\ 2s=53\times 18=954,$$

$$\therefore\ s=\frac{954}{2}=477.$$

Or thus by the formula.

Here $a=1$, $d=3$, and $n=18$; then

$$s=(2a+\overline{n-1}d)\frac{n}{2}=(2\times 1+17\times 3)\frac{18}{2}=477.$$

3. Here the 1st term$=5$, the common difference$=4$, and the 30th term$=5+(30-1)4=121$; hence the series may be written down,

$$s=5+9+13+\ \ldots\ +113+117+121,$$

writing the terms in an inverse order,

$$s=121+117+113+\ \ldots\ +13+9+5.$$

Adding the corresponding terms together,

$$2s=126+126+126+\ \ldots\ +126+126+126.$$

But there are 30 terms in this series,

$$\therefore\ 2s=126\times 30=3780,$$

$$\therefore\ s=\frac{3780}{2}=1890.$$

Or thus by the formula.

Here $a=5$, $d=4$, and $n=30$; then

$$s=(2a+\overline{n-1}d)\frac{n}{2}=(2\times 5+29\times 4)\frac{30}{2}=1890.$$

4. Here the 1st term$=1$, the common difference$=9$, and the nth term$=1+\overline{n-1}\times 9=9n-8$;

$$\therefore\ \text{sum}=(\text{1st term}+\text{last term})\times\frac{n}{2}=(1+9n-8)\frac{n}{2}=\frac{9n^2-7n}{2}.$$

5. $$3x^2+x=80,$$
$$\therefore\ x^2+\tfrac{1}{3}x=\tfrac{80}{3}.$$

Completing the square by adding $(\frac{1}{6})^2$ or $\frac{1}{36}$,

$$x^2+\tfrac{1}{3}x+(\tfrac{1}{6})^2=\tfrac{80}{3}+\tfrac{1}{36}=\tfrac{961}{36},$$
$$\therefore\ x+\tfrac{1}{6}=\sqrt{\tfrac{961}{36}}=\tfrac{31}{6},$$
$$\therefore\ x=\tfrac{31}{6}-\tfrac{1}{6}=\tfrac{30}{6}=5.$$

6. Let x=the number of terms; then, by formula (2.), we find the sum of the series by putting $a=1$, $d=1$, and x for n.

$$(2+\overline{x-1}\times1)\frac{x}{2}=36,$$
$$\therefore\ (x+1)\frac{x}{2}=36,$$
$$\therefore\ x^2+x=72.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$x^2+x+\tfrac{1}{4}=72+\tfrac{1}{4}=\tfrac{289}{4},$$
$$\therefore\ x+\tfrac{1}{2}=\sqrt{\tfrac{289}{4}}=\tfrac{17}{2},$$
$$\therefore\ x=\tfrac{17}{2}-\tfrac{1}{2}=\tfrac{16}{2}=8.$$

7. Here the 1st term=1, the common difference=1, the number of terms=36, and the last term=36; hence the series may be written

$$s=1+2+3+\ .\ .\ .\ .\ +34+35+36,$$
$$\text{and } s=36+35+34+\ .\ .\ .\ .\ +3+2+1,$$
$$\therefore\ 2s=37+37+37+\ .\ .\ .\ .\ +37+37+37,$$
$$=37\times36,$$
$$\therefore\ s=37\times\tfrac{36}{2}=37\times18=666.$$

Or thus by the formula.

Here $a=1$, $d=1$, and $n=36$; then

$$s=(2a+\overline{n-1}d)\frac{n}{2}=(2\times1+35\times1)\frac{36}{2}=37\times18=666.$$

52. GEOMETRICAL SERIES.

EXAMPLES.

1. Here the 1st term $=1$, the common ratio $=4$; $\therefore$ the 6th term $=1\times4^5=1024$, and the 7th term $=1024\times4=4096$; hence the series may be written

$$s=1+4+16+\ .\ .\ .\ .\ +1024+4096.$$

Multiplying by the common ratio 4,

$$4s=4+16+\ .\ .\ .\ .\ +4096+16384.$$

Here the first 6 terms of this latter series are the same as the last 6 terms of the former series; hence we find by subtraction,

$$3s=16384-1=16383,$$

$$\therefore\ s=\tfrac{16383}{3}=5461.$$

Or thus by formula (1.)

Here $a=1$, $r=4$, and $n=7$; then

$$s=\frac{a(r^n-1)}{r-1}=\frac{1(4^7-1)}{4-1}=\frac{16384-1}{3}=5461.$$

2. Here the 1st term $=1$, the common ratio $=3$; the 7th term $=1\times3^6=729$, and the 8th term $=729\times3=2187$; hence the series may be written

$$s=1+3+9+\ .\ .\ .\ .\ +729+2187.$$

Multiplying by the common ratio 3,

$$3s=3+9+\ .\ .\ .\ .\ +2187+6561.$$

Here the first 7 terms of this latter series are the same as the last 7 terms of the former series; hence we find by subtraction,

$$2s=6561-1=6560$$

$$\therefore\ s=\tfrac{6560}{2}=3280.$$

Or thus by formula (1.)

Here $a=1$, $r=3$, and $n=8$; then

$$s=\frac{a(r^n-1)}{r-1}=\frac{1(3^8-1)}{3-1}=\frac{6561-1}{2}=3280.$$

3. Here writing down all the terms of the series, we have

$$s=1+10+100+1000+10000,$$

$$\therefore\ 10s=10+100+1000+10000+100000,$$

$$\therefore\ 9s=100000-1=99999,$$

$$\therefore\ s=\frac{99999}{9}=11111.$$

Or thus by formula (1.)

Here $a=1$, $r=10$, and $n=5$; then

$$s=\frac{a(r^n-1)}{r-1}=\frac{1\,(10^5-1)}{10-1}=\frac{100000-1}{9}=11111.$$

4. Here the common ratio $=\frac{1}{3}$; then

$$s=\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{81}+\text{\&c. to infinity.}$$

Multiplying every term by the common ratio,

$$\tfrac{1}{3}s=\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{81}+\tfrac{1}{243}+\text{\&c. to infinity.}$$

Here the terms in the latter series are the same as the terms in the former with the exception of the 1st term; hence we find by subtracting the latter from the former,

$$\tfrac{2}{3}s=\tfrac{1}{3},\ \therefore\ s=\tfrac{1}{3}\times\tfrac{3}{2}=\tfrac{1}{2}.$$

Or thus by formula (3.)

Here $a=\frac{1}{3}$, $r=\frac{1}{3}$, n being infinite; then

$$s=\frac{a}{1-r}=\frac{\frac{1}{3}}{1-\frac{1}{3}}=\frac{1}{3-1}=\tfrac{1}{2}.$$

5. Here the common ratio $=\frac{1}{4}$; then

$$s=1+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\text{\&c. to infinity.}$$

Multiplying every term by $\frac{1}{4}$,

$$\tfrac{1}{4}s=\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{64}+\text{\&c. to infinity.}$$

Subtracting the latter from the former,

$$\tfrac{3}{4}s=1,\ \therefore\ s=\tfrac{4}{3}=1\tfrac{1}{3}.$$

Or thus by formula (3.)

Here $a=1$, $r=\frac{1}{4}$, and n infinite; then

$$s=\frac{a}{1-r}=\frac{1}{1-\frac{1}{4}}=\frac{4}{4-1}=\frac{4}{3}=1\frac{1}{3}.$$

7. Here the common ratio $=\frac{1}{100}$; then

$$s=\cdot 252525+\text{to infinity.}$$

Multiplying each side by 100,

$$100s=25\cdot 2525 \text{ to infinity.}$$

Subtracting the former from the latter,

$$99s=25, \therefore\ s=\frac{25}{99}.$$

Or thus by formula (3.)

Here $a=\frac{25}{100}$, $r=\frac{1}{100}$; then

$$s=\frac{a}{1-r}=\frac{\frac{25}{100}}{1-\frac{1}{100}}=\frac{25}{100-1}=\frac{25}{99}.$$

8. In this case the common ratio is 2; then

$$s=2+2^2+2^3+\ \ldots\ldots\ +2^{n-1}+2^n,$$

Multiplying every term by 2,

$$2s=2^2+2^3+\ \ldots\ldots\ +2^n+2^{n+1}.$$

Subtracting the former from the latter,

$$s=2^{n+1}-2.$$

Or thus by formula (1.)

Here $a=2$, $r=2$, and $n=n$; then

$$s=\frac{a(r^n-1)}{r-1}=\frac{2(2^n-1)}{2-1}=2(2^n-1)=2^{n+1}-2.$$

Art. 53.

2. Here every term of the series may be decomposed into two simple fractions, as follows: —

$\frac{1}{1.3}=\frac{1}{2}(\frac{1}{1}-\frac{1}{3})$; $\frac{1}{3.5}=\frac{1}{2}(\frac{1}{3}-\frac{1}{5})$; $\frac{1}{5.7}=\frac{1}{2}(\frac{1}{5}-\frac{1}{7})$;
$\frac{1}{7\cdot 9}=\frac{1}{2}(\frac{1}{7}-\frac{1}{9})$; and so on to infinity.

Hence we have, writing down the proposed series,

$$s=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\text{ &c.}$$

$$\therefore\ s=\tfrac{1}{2}(\tfrac{1}{1}-\tfrac{1}{3})+\tfrac{1}{2}(\tfrac{1}{3}-\tfrac{1}{5})+\tfrac{1}{2}(\tfrac{1}{5}-\tfrac{1}{7})+\tfrac{1}{2}(\tfrac{1}{7}-\tfrac{1}{9})+\text{ &c.}$$

Multiplying every term by 2,

$$2s=\overline{\tfrac{1}{1}-}\tfrac{1}{3}+\overline{\tfrac{1}{3}-}\tfrac{1}{5}+\overline{\tfrac{1}{5}-}\tfrac{1}{7}+\overline{\tfrac{1}{7}-}\tfrac{1}{9}+\text{ &c.},$$

Where the terms after the first destroy each other,

$$\therefore\ 2s=1,\ \ \therefore\ s=\tfrac{1}{2}.$$

54. SIMPLE INTEREST.

Ex. 2. Here $P=50$, $I=5$, $n=2\frac{1}{2}$, and r is required; then by (1.), we have

$$\frac{2\frac{1}{2}\times r\times 50}{100}=5,\ \therefore\ r=\frac{5\times 100}{2\frac{1}{2}\times 50}=4.$$

55. DISCOUNT.

Ex. 2. Here $D=30$, $M=530$, $n=2$, and r is required; then by formula (1.), we have

$$\frac{2\times r\times 530}{100+2\times r}=30,$$

$$\therefore\ 1060r=30(100+2r)=3000+60r,$$

$$\therefore\ 1060r-60r=3000,$$

$$\therefore\ 1000r=3000,\ \therefore\ r=3.$$

56. LOSS AND GAIN.

Ex. 3. Let x = the cost price, in pounds;
then by the data of the question,

the *loss*, in *pounds* $=\dfrac{8\times x}{100}$;

but we have also

the *loss* in *pounds* $= x-46$.

Hence by equality, we have

$$x-46=\frac{8x}{100},$$

$$\therefore\ 100x-4600=8x,$$

$$\therefore\ 92x=4600,$$

$$\therefore\ x=\tfrac{4600}{92}=£50.$$

$$x+\frac{x^2}{100}=39,$$

$$\therefore\ x^2+100x=3900.$$

Completing the square by adding 50^2 or 2500 to each side,

$$x^2+100x+50^2=3900+2500=6400,$$

$$\therefore\ x+50=\sqrt{6400}=80,$$

$$\therefore\ x=80-50=£30.$$

5. Let x=the cost price of the horse, in pounds; then by the question, we have

$$\text{Gain on } £100=2x,$$

$$\therefore\quad ,,\quad £1=\frac{2x}{100},$$

$$\therefore\quad ,,\quad £x=x \text{ times } \frac{2x}{100}=\frac{2x^2}{100};$$

$$\therefore \text{selling price, in pounds}=x+\frac{2x^2}{100};$$

but, by the question, the selling price was 12 pounds;

$$\therefore x+\frac{2x^2}{100}=12;$$

$$\therefore\ x^2+50x=600.$$

Completing the square by adding 25^2 or 625 to each side,

$$x^2+50x+25^2=600+625=1225,$$

$$\therefore\ x+25=\sqrt{1225}=35,$$

$$\therefore\ x=35-25=£10.$$

60. CUBIC EQUATIONS.

2. $$x^3+9x=6.$$

Here $\frac{1}{3}$ of $-9=-3$, then substituting $z-\frac{3}{z}$ for x, the equation becomes

$$\left(z-\frac{3}{z}\right)^3+9\left(z-\frac{3}{z}\right)=6.$$

Expanding the binomial (Art. **35.**, Ex. 9), &c., we get

$$z^3-9z+\frac{27}{z}-\frac{27}{z^3}+9z-\frac{27}{z}=6.$$

Observing that certain terms destroy each other, we get

$$z^3-\frac{27}{z^3}=6,$$

$$\therefore\ z^6-27=6z^3,$$

$$\therefore\ z^6-6z^3=27,$$

which may be solved as a quadratic (see **Art. 42.**, **Ex. 25**); therefore completing the square by adding 3^2 or 9 to each side,

$$z^6-6z^3+9=27+9=36,$$

$$\therefore\ z^3-3=6,$$

$$\therefore\ z^3=6+3=9,$$

$$\therefore\ z=\sqrt[3]{9}=2\cdot080084.$$

For the value of x, we have

$$x=z-\frac{3}{z}=2\cdot080084-\frac{3}{2\cdot080084}=\cdot637834.$$

3. $$x^3+12x=12.$$

Here $\frac{1}{3}$ of $-12=-4$, then substituting $z-\frac{4}{z}$ for x, the equation becomes,

$$\left(z-\frac{4}{z}\right)^3+12\left(z-\frac{4}{z}\right)=12.$$

Expanding the binomial, we get

$$z^3-12z+\frac{48}{z}-\frac{64}{z^3}+12z-\frac{48}{z}=12,$$

$$\therefore\ z^3-\frac{64}{z^3}=12,$$

$$\therefore\ z^6-64=12z^3,$$

$$\therefore\ z^6-12z^3=64.$$

Completing the square by adding 6^2 or 36,

$$z^6-12z^3+6^2=64+36=100,$$
$$\therefore\ z^3-6=10,$$
$$\therefore\ z^3=10+6=16,$$
$$\therefore\ z=\sqrt[3]{16}=2{\cdot}519842.$$

For the value of x, we have

$$x=z-\frac{4}{z}=2{\cdot}519842-\frac{4}{2{\cdot}519842}={\cdot}932441.$$

4. $$x^3+6x=7.$$

Here $\frac{1}{3}$ of $-6=-2$, then substituting $z-\frac{2}{z}$ for x, the equation becomes,

$$\left(z-\frac{2}{z}\right)^3+6\left(z-\frac{2}{z}\right)=7.$$

Expanding the binomial, we get

$$z^3-6z+\frac{12}{z}-\frac{8}{z^3}+6z-\frac{12}{z}=7,$$
$$\therefore\ z^3-\frac{8}{z^3}=7,$$
$$\therefore\ z^6-8=7z^3,$$
$$\therefore\ z^6-7z^3=8.$$

Completing the square by adding $(\frac{7}{2})^2$ or $\frac{49}{4}$ to each side,

$$z^6-7z^3+(\tfrac{7}{2})^2=8+\tfrac{49}{4}=\tfrac{81}{4},$$
$$\therefore\ z^3-\tfrac{7}{2}=\tfrac{9}{2},$$
$$\therefore\ z^3=\tfrac{9}{2}+\tfrac{7}{2}=8,$$
$$\therefore\ z=\sqrt[3]{8}=2.$$

For the value of x, we have

$$x=z-\frac{2}{z}=2-\frac{2}{2}=2-1=1.$$

5. $$x^3+9x=-6.$$

Here $\frac{1}{3}$ of $-9=-3$, then substituting $z-\frac{3}{z}$ for x, the proposed equation becomes

$$\left(z-\frac{3}{z}\right)^3+9\left(z-\frac{3}{z}\right)=-6.$$

Expanding the binomial, we get

$$z^3-9z+\frac{27}{z}-\frac{27}{z^3}+9z-\frac{27}{z}=-6,$$

$$\therefore\ z^3-\frac{27}{z^3}=-6,$$

$$\therefore\ z^6-27=-6z^3,$$

$$\therefore\ z^6+6z^3=\ \ 27.$$

Completing the square by adding 3^2 or 9 to each side,

$$z^6+6z^3+9=27+9=36,$$

$$\therefore\ z^3+3=6,$$

$$\therefore\ z^3=6-3=3,$$

$$\therefore\ z=\sqrt[3]{3}=1{\cdot}44225.$$

And for the value of x, we have

$$x=z-\frac{3}{z}=1{\cdot}44225-\frac{3}{1{\cdot}44225}=1{\cdot}44225-2{\cdot}08008$$

$$=-{\cdot}63783.$$

6. $$x^3+15x=20.$$

Here $\frac{1}{3}$ of $-15=-5$, then substituting $z-\frac{5}{z}$ for x, the proposed equation becomes

$$\left(z-\frac{5}{z}\right)^3+15\left(z-\frac{5}{z}\right)=20,$$

Expanding the binomial, we get

$$z^3-15z+\frac{75}{z}-\frac{125}{z^3}+15z-\frac{75}{z}=20,$$

$$\therefore\ z^3-\frac{125}{z^3}=20,$$

$$\therefore\ z^6-125=20z^3,$$

$$\therefore\ z^6-20z^3=125.$$

Completing the square by adding 10^2 or 100 to each side,

$$z^6-20z^3+10^2=125+100=225,$$

$$\therefore\ z^3-10=15,$$

$$\therefore\ z^3=15+10=25,$$

$$\therefore\ z=\sqrt[3]{25}=2{\cdot}924018.$$

And for the value of x, we have

$$x=z-\frac{5}{z}=2{\cdot}924018-\frac{5}{2{\cdot}924018}=2{\cdot}924018-1{\cdot}709976$$
$$=1{\cdot}214042.$$

7. $$x^3-3x=2.$$

Here $\frac{1}{3}$ of $+3=1$, then substituting $z+\frac{1}{z}$ for x, the proposed equation becomes

$$\left(z+\frac{1}{z}\right)^3+3\left(z+\frac{1}{z}\right)=2.$$

Expanding the binomial, and reducing,

$$z^3+3z+\frac{3}{z}+\frac{1}{z^3}-3z-\frac{3}{z}=2,$$
$$\therefore\ z^3+\frac{1}{z^3}=2.$$
$$\therefore\ z^6+1=2z^3,$$
$$\therefore\ z^6-2z^3=-1.$$

Completing the square by adding 1^2 or 1 to each side,

$$z^6-2z^3+1=-1+1=0,$$
$$\therefore\ z^3-1=\ 0,$$
$$\therefore\ z^3=1,$$
$$\therefore\ z=\sqrt[3]{1}=1.$$

And for the value of x, we have

$$x=z+\frac{1}{z}=1+\frac{1}{1}=1+1=2.$$

8. $$x^3-27x=-54.$$

Here $\frac{1}{3}$ of $+27=9$, then substituting $z+\frac{9}{z}$ for x, the proposed equation becomes

$$\left(z+\frac{9}{z}\right)^3-27\left(z+\frac{9}{z}\right)=-54.$$

Expanding the binomial and reducing,

$$z^3+27z+\frac{243}{z}+\frac{729}{z^3}-27z-\frac{243}{z}=-54,$$

$$\therefore\ z^3+\frac{729}{z^3}=-54,$$

$$\therefore\ z^6+729=-54z^3,$$

$$\therefore\ z^6+54z^3=-729.$$

Completing the square by adding 27^2 or 729, or what amounts to the same thing in this case, transposing -729,

$$z^6+54z^3+729=-729+729=0.$$

Taking the sq. root,

$$z^3+27=0,$$

$$\therefore\ z^3=-27 \text{ or } (-3)^3,$$

$$\therefore\ z=-3.$$

Substituting this value of z, we have

$$x=z+\frac{9}{z}=-3+\frac{9}{-3}=-3-3=-6.$$

Art. 61.

2. $$y^3+3y^2+24y=20.$$

Here we must first remove the second term of the equation. For this purpose, we have

$$\tfrac{1}{3} \text{ of } -3=-1.$$

Then substituting $x-1$ for y, in the right-hand number of the proposed equation, we find

$$\begin{aligned}&y^3+3y^2+24y\\&=(x-1)^3+3(x-1)^2+24(x-1), \text{ by expansion}\\&=x^3-3x^2+3x-1+3x^2-6x+3+24x-24\\&=x^3+21x-22.\end{aligned}$$

$$\therefore\ x^3+21x-22=20,$$

$$\therefore\ x^3+21x=20+22=42.$$

We can now apply the foregoing method for solving a cubic equation.

Here $\frac{1}{3}$ of $-21=-7$, then substituting $z-\frac{7}{z}$ for x, the last equation becomes

$$\left(z-\frac{7}{z}\right)^3+21\left(z-\frac{7}{z}\right)=42.$$

By developing the binomial, &c.,

$$z^3-21z+\frac{147}{z}-\frac{343}{z^3}+21z-\frac{147}{z}=42,$$

$$\therefore\ z^3-\frac{343}{z^3}=42,$$

$$\therefore\ z^6-343=42z^3,$$

$$\therefore\ z^6-42z^3=343.$$

Adding 21^2 or 441 to each side to complete the square,

$$z^6-42z^3+21^2=343+441=784.$$

Taking the square root,

$$\therefore\ z^3-21=\sqrt{784}=28,$$

$$\therefore\ z^3=28+21=49,$$

$$\therefore\ z=\sqrt[3]{49}=3{\cdot}659306.$$

And to find the value of x, we have

$$x=z-\frac{7}{z}=3{\cdot}659306-\frac{7}{3{\cdot}659306}=3{\cdot}659306-1{\cdot}912931$$
$$=1{\cdot}746375.$$

And to find the value of y in the original equation, we have

$$y=x-1=1{\cdot}746375-1={\cdot}746375.$$

ART. 62.

2. $$x^3+4x=16.$$

First multiplying by x,

$$x^4+4x^2=16x,$$

adding $4x^2$ to each side,

$$x^4+8x^2=4x^2+16x,$$

adding 16 to each side to complete the square,

$$x^4+8x^2+16=4x^2+16x+16,$$

or

$$(x^2+4)^2=(2x+4)^2$$
$$\therefore\ x^2+4=2x+4,$$
$$\therefore\ x^2=2x,$$
$$\therefore\ x=2.$$

3. $$x^3-8x=3.$$

First multiplying by x,

$$x^4-8x^2=3x,$$

adding $9x^2$ to each side,

$$x^4+x^2=9x^2+3x,$$

adding $\frac{1}{4}$ to each side to complete the square,

$$x^4+x^2+\tfrac{1}{4}=9x^2+3x+\tfrac{1}{4},$$

or

$$(x^2+\tfrac{1}{2})^2=(3x+\tfrac{1}{2})^2,$$
$$\therefore\ x^2+\tfrac{1}{2}=3x+\tfrac{1}{2},$$
$$\therefore\ x^2=3x,$$
$$\therefore\ x=3.$$

4. $$x^3-7x=-6\ .\ .\ .\ .\ (1.)$$

First multiplying by x,

$$x^4-7x^2=-6x,$$

adding $9x^2$ to each side,

$$x^4+2x^2=9x^2-6x,$$

adding one to each side to complete the square,

$$x^4+2x^2+1=9x^2-6x+1,$$
$$\text{or } (x^2+1)^2=(3x-1)^2,$$
$$\therefore\ x^2+1=3x-1,$$
$$\therefore\ x^2-3x=-2\ .\ .\ .\ .\ (2.)$$

adding $(\frac{3}{2})^2$ or $\frac{9}{4}$ to each side to complete the square,

$$x^2-3x+(\tfrac{3}{2})^2=-2+\tfrac{9}{4}=\tfrac{1}{4},$$
$$\text{or } (x-\tfrac{3}{2})^2=\tfrac{1}{4} \text{ or } (\pm\tfrac{1}{2})^2,$$
$$\therefore\ x-\tfrac{3}{2}=\tfrac{1}{2} \text{ or } -\tfrac{1}{2}$$
$$\therefore\ x=2 \text{ or } 1.$$

Here we have found two roots of the given equation, (1.); in order to find the remaining root, transpose the constants of eq. (1.) and eq. (2.); then

$$x^3-7x+6=0,$$
$$\text{and } x^2-3x+2=0.$$

Dividing the former by the latter,

$$\frac{x^3-7x+6}{x^2-3x+2}=0.$$

By actual division, as shown below,* we get

$$x+3=\quad 0,$$
$$\therefore\ x=-3.$$

So that the roots of the given equation are 1, 2, and -3.

$$\begin{array}{l}
{}^{*}x^2-3x+2)\,x^3-7x+6\ (x+3 \\
\qquad\qquad\quad \underline{x^3-3x^2+2x} \\
\qquad\qquad\qquad\quad 3x^2-9x+6 \\
\qquad\qquad\qquad\quad \underline{3x^2-9x+6} \\
\qquad\qquad\qquad\qquad * \quad * \quad *
\end{array}$$

6. $$x^4-2x^3+x=30.$$

Adding and subtracting x^2,

$$x^4-2x^3+x^2-(x^2-x)=30,$$
$$\therefore\ (x^2-x)^2-(x^2-x)=30.$$

This equation may be solved as a quadratic for the value of x^2-x. Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$(x^2-x)^2-(x^2-x)+\tfrac{1}{4}=30+\tfrac{1}{4}=\tfrac{121}{4},$$
$$\text{or } \{(x^2-x)-\tfrac{1}{2}\}^2=\tfrac{121}{4} \text{ or } (\pm\tfrac{11}{2})^2,$$
$$\therefore\ x^2-x-\tfrac{1}{2}=\tfrac{11}{2} \text{ or } -\tfrac{11}{2},$$
$$\therefore\ x^2-x=6 \text{ or } -5.$$

Here we have two equations, viz., $x^2-x=6$, and $x^2-x=-5$: first let us take the equation,

$$x^2-x=6.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$x^2-x+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4},$$
$$\text{or } (x-\tfrac{1}{2})^2=\tfrac{25}{4} \text{ or } (\pm\tfrac{5}{2})^2,$$
$$\therefore\ x-\tfrac{1}{2}=\tfrac{5}{2} \text{ or } -\tfrac{5}{2},$$
$$\therefore\ x=3 \text{ or } -2,$$

which are the roots given in the text.

Again, taking the equation,

$$x^2-x=-5,$$
$$\therefore\ x^2-x+\tfrac{1}{4}=-5+\tfrac{1}{4}=\tfrac{19}{4},$$
$$\therefore\ x-\tfrac{1}{2}=\pm\sqrt{\frac{19}{4}}=\pm\frac{\sqrt{19}}{2},$$
$$\therefore\ x=\tfrac{1}{2}\pm\frac{\sqrt{19}}{2}=\tfrac{1}{2}(1\pm\sqrt{19}).$$

7. $x^{4n}+2x^{3n}-x^n=2$.

Adding and subtracting x^{2n},

$$x^{4n}+2x^{3n}+x^{2n}-x^{2n}-x^n=2$$
$$(x^{2n}+x^n)^2-(x^{2n}+x^n)=2.$$

Solving this as a quadratic for the value of $x^{2n}+x^n$; completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side of the equation,

$$(x^{2n}+x^n)^2-(x^{2n}+x^n)+\tfrac{1}{4}=2+\tfrac{1}{4}=\tfrac{9}{4},$$
$$\text{or}\qquad \{(x^{2n}+x^n)-\tfrac{1}{2}\}^2=\tfrac{9}{4} \text{ or } (\pm\tfrac{3}{2})^2,$$
$$\therefore\ x^{2n}+x^n-\tfrac{1}{2}=\tfrac{3}{2} \text{ or} -\tfrac{3}{2},$$
$$\therefore\ x^{2n}+x^n=2 \text{ or } -1.$$

Here we have two equations, viz., $x^{2n}+x^n=2$, and $x^{2n}+x^n=-1$, first taking the equation

$$x^{2n}+x^n=2.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$x^{2n}+x^n+\tfrac{1}{4}=2+\tfrac{1}{4}=\tfrac{9}{4},$$
$$\text{or } (x^n+\tfrac{1}{2})^2=\tfrac{9}{4} \text{ or } (\pm\tfrac{3}{2})^2,$$
$$\therefore\ x^n+\tfrac{1}{2}=\tfrac{3}{2} \text{ or } -\tfrac{3}{2},$$
$$\therefore\ x^n=1 \text{ or} -2.$$

Taking the nth root $\therefore\ x=1$ or $\sqrt[n]{-2}$.

These are the roots given in the text.

Again, taking the equation

$$x^{2n}+x^n=-1.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$,

$$x^{2n}+x^n+\tfrac{1}{4}=-1+\tfrac{1}{4}=-\tfrac{3}{4},$$
$$\therefore\ x^n+\tfrac{1}{2}=+\sqrt{-\tfrac{3}{4}}=+\frac{\sqrt{-3}}{2},$$
$$\therefore\ x^n=\pm\frac{\sqrt{-3}}{2}-\tfrac{1}{2}=\tfrac{1}{2}(\pm\sqrt{-3}-1).$$

Taking the nth root of each side,

$$x=\{\tfrac{1}{2}(+\sqrt{-3}-1)\}^{\frac{1}{n}}.$$

8. $\left(x+\dfrac{1}{x}\right)^2+6\left(x+\dfrac{1}{x}\right)=\frac{85}{4}$.

Adding 3^2 or 9 to each side to complete the square,

$$\left(x+\frac{1}{x}\right)^2+6\left(x+\frac{1}{x}\right)+9=\tfrac{85}{4}+9=\tfrac{121}{4}.$$

Taking the square root of each side

$$\overline{x+\frac{1}{x}+3}=\tfrac{11}{2} \text{ or} -\tfrac{11}{2},$$
$$\therefore\ x+\frac{1}{x}=\tfrac{5}{2} \text{ or } -\tfrac{17}{2}.$$

First taking the equation

$$x+\frac{1}{x}=\frac{5}{2},$$
$$\therefore\ x^2+1=\tfrac{5}{2}x,$$
$$\therefore\ x^2-\tfrac{5}{2}x=-1,$$

Adding $(\frac{5}{4})^2$ or $\frac{25}{16}$ to each side to complete the square,

$$x^2-\frac{5}{2}x+(\frac{5}{4})^2=-1+\frac{25}{16}=\frac{9}{16},$$
$$\text{or } (x-\frac{5}{4})^2=\frac{9}{16} \text{ or } (\pm\frac{3}{4})^2,$$
$$\therefore\ x-\frac{5}{4}=\frac{3}{4} \text{ or } -\frac{3}{4},$$
$$\therefore\ x=2 \text{ or } \frac{1}{2}.$$

These are the roots given in the text.

Second taking the equation

$$x+\frac{1}{x}=-\frac{17}{2},$$
$$\therefore\ x^2+1=-\frac{17}{2}x,$$
$$\therefore\ x^2+\frac{17}{2}x=-1.$$

Completing the square by adding $(\frac{17}{4})^2$ or $\frac{289}{16}$ to each side,

$$x^2+\frac{17}{2}x+(\frac{17}{4})^2=-1+\frac{289}{16}=\frac{273}{16}$$
$$\therefore\ x+\frac{17}{4}=\pm\sqrt{\frac{273}{16}}=\pm\frac{\sqrt{273}}{4}$$
$$\therefore\ x=\pm\frac{\sqrt{273}}{4}-\frac{17}{4}=\frac{1}{4}(\pm\sqrt{273}-17),$$

which gives an expression for the two other roots.

9. $$x^4+2x^3+2x+1=\frac{142x^2}{9}.$$

Dividing every term by x^2,

$$x^2+2x+\frac{2}{x}+\frac{1}{x^2}=\frac{142}{9},$$
$$\therefore\ x^2+\frac{1}{x^2}+2\ (x+\frac{1}{x})=\frac{142}{9}.$$

Adding 2 to each side in order to render $x^2+\frac{1}{x^2}$ a square quantity,

$$x^2+2+\frac{1}{x^2}+2\ (x+\frac{1}{x})=\frac{142}{9}+2=\frac{160}{9},$$
$$\text{or } (x+\frac{1}{x})^2+2\ (x+\frac{1}{x})=\frac{160}{9}.$$

Solving this equation as a quadratic for $x+\frac{1}{x}$, by adding 1 to each side to complete the square,

$$(x+\frac{1}{x})^2+2\,(x+\frac{1}{x})+1=\frac{160}{9}+1=\frac{169}{9},$$

$$\text{or } (\overline{x+\frac{1}{x}+1})^2=\frac{169}{9} \text{ or } (\pm\frac{13}{3})^2,$$

$$\therefore\ x+\frac{1}{x}+1=\frac{13}{3} \text{ or } -\frac{13}{3},$$

$$\therefore\ x+\frac{1}{x}=\frac{10}{3} \text{ or } -\frac{16}{3}.$$

First taking the equation

$$x+\frac{1}{x}=\frac{10}{3},$$

$$\therefore\ x^2+1=\frac{10}{3}x,$$

$$\therefore\ x^2-\frac{10}{3}x=-1.$$

Completing the square by adding $(\frac{5}{3})^2$ or $\frac{25}{9}$ to each side,

$$x^2-\frac{10}{3}+(\frac{5}{3})^2=-1+\frac{25}{9}=\frac{16}{9},$$

$$\therefore\ x-\frac{5}{3}=\frac{4}{3} \text{ or} -\frac{4}{3},$$

$$\therefore\ x=3 \text{ or } \frac{1}{3}.$$

Second, taking the equation

$$x+\frac{1}{x}=-\frac{16}{3}$$

$$\therefore\ x^2+1=-\frac{16}{3}x$$

$$\therefore\ x^2+\frac{16}{3}x=-1.$$

Completing the square by adding $(\frac{8}{3})^2$ or $\frac{64}{9}$ to each side,

$$x^2+\frac{16}{3}x+(\frac{8}{3})^2=-1+\frac{64}{9}=\frac{55}{9},$$

$$\therefore\ x+\frac{8}{3}=\pm\sqrt{\frac{55}{9}}=\pm\frac{\sqrt{55}}{3},$$

$$\therefore\quad x=\pm\frac{\sqrt{55}}{3}-\frac{8}{3}=\frac{1}{3}\,(\pm\sqrt{55}-8).$$

11. $(x^{\frac{1}{3}}+4)^{\frac{1}{2}}-(x^{\frac{1}{3}}-4)^{\frac{1}{2}}=2 \ .\ .\ .\ .\ (1.)$

Multiply each side of the equation by $(x^{\frac{1}{3}}+4)^{\frac{1}{2}}+(x^{\frac{1}{3}}-4)^{\frac{1}{2}}$, in order to have the product of the sum by the difference on the left hand side of the equation; then

$$\{(x^{\frac{1}{2}}+4)^{\frac{1}{2}}-(x^{\frac{1}{2}}-4)^{\frac{1}{2}}\}\times\{(x^{\frac{1}{2}}+4)+(x^{\frac{1}{2}}-4)^{\frac{1}{2}}\}$$
$$=2\{(x^{\frac{1}{2}}+4)^{\frac{1}{2}}+(x^{\frac{1}{2}}-4)^{\frac{1}{2}}\};$$

then by Art. **34**, or by actual multiplication, we get

$$(x^{\frac{1}{2}}+4)-(x^{\frac{1}{2}}-4)=2\{(x^{\frac{1}{2}}+4)^{\frac{1}{2}}+(x^{\frac{1}{2}}-4)^{\frac{1}{2}}\};$$
$$\therefore \quad 8=2\{(x^{\frac{1}{2}}+4)^{\frac{1}{2}}+(x^{\frac{1}{2}}-4)^{\frac{1}{2}}\};$$
$$\therefore (x^{\frac{1}{2}}+4)^{\frac{1}{2}}+(x^{\frac{1}{2}}-4)^{\frac{1}{2}}=4.$$

Adding this equation to the given eq. (1.), we get

$$2(x^{\frac{1}{2}}+4)^{\frac{1}{2}}=6,$$
$$\therefore (x^{\frac{1}{2}}+4)^{\frac{1}{2}}=3.$$

Squaring each side,

$$\therefore x^{\frac{1}{2}}+4=9,$$
$$\therefore x^{\frac{1}{2}}=5, \therefore x=25.$$

12. $$x(x+1)=6,$$
$$\therefore x^2+x=6.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$x^2+x+\tfrac{1}{4}=6+\tfrac{1}{4}=\tfrac{25}{4},$$
$$\therefore x+\tfrac{1}{2}=\tfrac{5}{2} \text{ or } -\tfrac{5}{2},$$
$$\therefore x=2 \text{ or } -3.$$

13. $$x^3-19x=30.$$

Adding $15x$ to each side, we get

$$x^3-4x=15x+30,$$
$$\therefore x(x^2-4)=15(x+2),$$

by Art. **32**,

$$x(x+2)(x-2)=15(x+2).$$

Dividing each side by the factor $x+2$,

$$x(x-2)=15,$$
$$\therefore x^2-2x=15.$$

Completing the square,

$$x^2-2x+1=16,$$
$$\therefore\ x-1=4 \text{ or } -4,$$
$$\therefore\ x=5 \text{ or } -3,$$

and from the factor $x+2$, we get the equation

$$x+2=0,\ \therefore\ x=-2.$$

So that the roots are 5 or -3 or -2.

14. $$x^3-76x=-240.$$

Adding $60x$ to each side,

$$x^3-16x=60x-240,$$
$$\therefore\ x(x^2-16)=60(x-4),$$

by Art. **32**, $$x(x+4)(x-4)=60(x-4).$$

Dividing by the common factor $x-4$,

$$x(x+4)=60,$$
$$\therefore\ x^2+4x=60.$$

Completing the sq., $$x^2+4x+4=64,$$
$$\therefore\ x+2=8 \text{ or } -8,$$
$$\therefore\ x=6 \text{ or } -10.$$

And from the factor $x-4$, we get

$$x-4=0,\ \therefore\ x=4.$$

So that the roots are 6 or -10 or 4.

APPLICATION OF ALGEBRA TO MENSURATION.

2. Let $6x$ = the base of the triangle, in feet; then
$5x$ = the perpendicular ,,

$\therefore$ the area of the triangle in sq. ft. $=\dfrac{6x\times 5x}{2}=15x^2$;

but, by the question, the area is 240 sq. ft.;

$$\therefore\ 15x^2=240,$$
$$\therefore\ x^2=16,\ \therefore\ x=\sqrt{16}=4;$$

and the base $=6x=6\times 4=24$ ft.

3. Let x=the breadth, in feet; then the area in sq. ft. $=20\times x$; but, by the question, the area is 30 sq. ft.;

$$\therefore\ 20x=30,$$

$$\therefore\ x=\tfrac{30}{20}=\tfrac{3}{2}\text{ ft.}=18\text{ in.}$$

4. Let x=the side of the square, in yards; then the area of the square, in sq. yds.$=x\times x=x^2$; but, by the question, the area is 84 sq. yds.;

$$\therefore\ x^2=84,\ \therefore\ x=\sqrt{84}=9{\cdot}165\text{ yds.}$$

5. Let x= the perpendicular, in feet; then

$$\text{the area, in sq. ft.}=\frac{4\times x}{2}=2x;$$

but, by the question, the area is 16 sq. ft.;

$$\therefore\ 2x=16,\ \therefore\ x=8\text{ ft.}$$

7. Let x= the base; then
$x-2$= the perpendicular;

$$\therefore\ x^2+(x-2)^2=10^2=100,$$
$$\therefore\ x^2+x^2-4x+4=100,$$
$$\therefore\ 2x^2-4x=100-4=96,$$
$$\therefore\ x^2-2x=48.$$

Completing the sq., $x^2-2x+1=49$,

$$\therefore\ (x-1)^2=49,$$
$$\therefore\ x-1=7,\ \therefore\ x=8,\text{ the base;}$$

and the perpendicular $=x-2=8-2=6$.

8. Let x=the side of the square, in feet; then

$$x^2+x^2=24^2=576,$$
$$\therefore\ 2x^2=576,$$
$$\therefore\ x^2=288,\ \therefore\ x=\sqrt{288}=16{\cdot}97\text{ ft.}$$

9. Let $x=$ the side of the inscribed square, in feet. Now as the diameter of the circle forms the diagonal of the square,

$$\text{the diagonal} = 4\text{ ft.} \times 2 = 8\text{ ft.};$$
$$\therefore\ x^2+x^2=8^2=64,$$
$$\therefore\ 2x^2=64,$$
$$\therefore\ x^2=32,\ \therefore\ x=\sqrt{32}=5{\cdot}65\text{ ft.}$$

10. $$x(x+6)=27,$$
$$\therefore\ x^2+6x=27.$$

Completing the sq. $$x^2+6x+9=36,$$
$$\therefore\ x+3=6,$$
$$\therefore\ x=3\text{ ft.}$$

11. Let $x=$ the one side; then
$7-x=$ the other side.

The area $= x\times(7-x)$; but, by the question, the area is 10;

$$\therefore\ x\times(7-x)=\ 10.$$
$$\therefore\ 7x-x^2=\ 10.$$

Changing all the signs $$x^2-7x=-10.$$

Completing the square by adding $(\frac{7}{2})^2$ or $\frac{49}{4}$,

$$x^2-7x+(\tfrac{7}{2})^2=-10+\tfrac{49}{4}=\tfrac{9}{4},$$
$$\therefore\ x-\tfrac{7}{2}=\tfrac{3}{2}\text{ or }-\tfrac{3}{2},$$
$$\therefore\ x=5\text{ or }2,$$
$$\therefore\ \text{the other side}=7-x=2\text{ or }5.$$

12. Let $x=$ the base; then
$18-x=$ the perpendicular.

The area $=\dfrac{x\times(18-x)}{2}$; but, by the question, the area is 40;

$$\therefore\ \frac{x\times(18-x)}{2}=40,$$
$$\therefore\ 18x-x^2=80.$$

Changing all the signs, $$x^2-18x=-80.$$

Completing the square by adding 9^2 or 81 to each side,

$$x^2-18x+81=-80+81=1$$
$$\therefore\ x-9=1,$$
$$\therefore\ x=10, \text{ the base.}$$

13. Let $x=$ the side of the square; then
The area $=x\times x=x^2$; but, by the question, we have also
The area $=2\ (4x)=8x$;

$$\therefore\ x^2=8x.$$

Dividing each side by x, we get

$$x=8, \text{ the side required.}$$

14. Let $x=$ the side of the cube; then
The content of the cube $=x\times x\times x=x^3$;
And the surface $=x\times x\times 6=6x^2$.
Therefore by the question, we have

$$x^3=12x^2,$$

dividing by x^2, $x=12$, the side required.

16. Let $x=$ the exterior diameter, in inches; then
$x-4=$ the interior „ „

The content of the material, in c. in.

$$=\{x^2\times{\cdot}7854-(x-4)^2\times{\cdot}7854\}\times 10\times 12.$$

But, by the question, we have also

The content of the material, in c. in. $=12\times 1728$,

$$\therefore\ \{x^2\times{\cdot}7854-(x-4)^2\times{\cdot}7854\}\times 10\times 12=12\times 1728,$$
$$\therefore\ x^2-(x-4)^2=\frac{12\times 1728}{{\cdot}7854\times 10\times 12}=220{\cdot}011,$$
$$\therefore\ x^2-x^2+8x-16=220{\cdot}011$$
$$\therefore\ 8x=236{\cdot}011,$$
$$\therefore\ x=\frac{236{\cdot}011}{8}=29{\cdot}501 \text{ inches.}$$

17. Let $x =$ the diameter; then
The surface of the 2 ends $= x^2 \times \cdot7854 \times 2$; and
The convex surface $= x \times 3\cdot1416 \times a$;
But, by the question, these are equal to each other,

$$\therefore\ x^2 \times \cdot7854 \times 2 = x \times 3\cdot1416 \times a.$$

First, dividing by ·7854, we get

$$x^2 \times 2 = 4ax$$
$$\therefore\ x = 2a.$$

19. Let $5x =$ the side AB; then
$4x =$ the side BC.

$$BD^2 = AB^2 - AD^2 = (5x)^2 - 12^2 = 25x^2 - 144;$$

also $BD^2 = BC^2 - DC^2 = (4x)^2 - 9^2 = 16x^2 - 81$;

therefore, by equality, we have

$$25x^2 - 144 = 16x^2 - 81,$$
$$\therefore\ 25x^2 - 16x^2 = 144 - 81 = 63,$$
$$\therefore\ 9x^2 = 63,$$
$$\therefore\ x^2 = \tfrac{63}{9} = 7,\ \therefore\ x = \sqrt{7} = 2\cdot6457:$$

$\therefore$ the side $AB = 5x = 2\cdot6457 \times 5 = 13\cdot228$,

and the side $BC = 4x = 2\cdot6457 \times 3 = 12\cdot583$.

20. Let ABC represent the triangle (see *fig.* to Prob. 18), BD the perpendicular, and AD and DC the segments of the base.

Let $x =$ the segment AD,
and $y =$ „ „ DC,

$$AB^2 = AD^2 + BD^2 = x^2 + 4^2 = x^2 + 16;$$
$$BC^2 = DC^2 + BD^2 = y^2 + 4^2 = y^2 + 16;$$
$$\therefore\ AB^2 + BC^2 = x^2 + y^2 + 32;$$

but, by the question, this is equal to 45,

$$\therefore\ x^2 + y^2 + 32 = 45,$$
$$\therefore\ x^2 + y^2 = 13 \quad .\ .\ .\ .\ (1).$$

But the base of the triangle is 5,

$$\therefore\ x + y = 5 \quad .\ .\ .\ .\ (2).$$

From eq. (1.) and eq. (2.), the values of x and y may be found.

Squaring eq. (2.) and subtracting eq. (1.) from the result, we get

$$2xy=25-13=12.$$

Subtracting this from eq. (1.), we get

$$x^2-2xy+y^2=1,$$

$$\therefore\ x-y=1\ \ldots\ldots\ (3.)$$

Adding this to eq. (2.), we get

$$2x=6,\ \therefore\ x=3.$$

Subtracting eq. (3.) from eq. (2.), we get

$$2y=4,\ \therefore\ y=2.$$

$$AB^2=x^2+16=3^2+16=25,\ \therefore\ AB=\sqrt{25}=5;$$

$$BC^2=y^2+16=2^2+16=20,\ \therefore\ BC=\sqrt{20}.$$

21. Here, we have (see the last example)

$$AB^2-BC^2=x^2-y^2;$$

but, by the question, this is equal to 15,

$$\therefore\ x^2-y^2=15\ \ldots\ldots\ (1.)$$

but the base of the triangle is 5,

$$\therefore\ x+y=5\ \ldots\ldots\ (2.)$$

Dividing eq. (1.) by eq. (2.), we get

$$x-y=3\ \ldots\ldots\ (3.)$$

Adding eq. (3.) and eq. (2.), we get

$$2x=8,\ \therefore\ x=4.$$

Subtracting eq. (3.) from eq. (2.), we get

$$2y=2,\ \therefore\ y=1.$$

$$AB^2=x^2+16=4^2+16=32,\ \therefore\ AB=\sqrt{32}=5{\cdot}656;$$

$$BC^2=y^2+16=1^2+16=17,\ \therefore\ BC=\sqrt{17}=4{\cdot}123.$$

22. Let x=the one side, and y=the other side.

First, from the property of the right-angled triangle, we have

$$x^2+y^2=50^2=2500\ \ldots\ldots\ (1.)$$

In order to obtain another equation, we have
the area of the rectangle$=xy$; but we have also,
the area of the rectangle$=$twice the area of the triangle
$=$the diagonal × the perpendicular
$=50\times 24=1200.$

Therefore by equality, we have

$$xy=1200 \quad .\ .\ .\ .\ (2.)$$

Adding twice eq. (2.) to eq. (1.), we get

$$x^2+2xy+y^2=2500+2400=4900,$$
$$\therefore\ x+y=70 \quad .\ .\ .\ .\ (3.)$$

Subtracting twice eq. (2.) from eq. (1.) we get

$$x^2-2xy+y^2=100,$$
$$\therefore\ x-y=10 \quad .\ .\ .\ .\ (4.)$$

Adding eq. (4.) and eq. (3.), we get

$$2x=80,\ \therefore\ x=40.$$

Subtracting eq. (4) from eq. (3.), we get

$$2y=60,\ \therefore\ y=30.$$

23. $$x^2=100+(x-6)^2.$$
$$\therefore\ x^2=100+x^2-12x+36,$$
$$\therefore\ 12x=136,\ \therefore\ x=\tfrac{136}{12}=11\tfrac{1}{3}\text{ ft.}$$

24. Let $x=$the height DC, in feet; then

$$\text{DO}=\text{OC}-\text{DC}=15-x;\ \text{AD}=\frac{\text{AB}}{2}=\frac{18}{2}=9;$$
$$\text{DO}^2=\text{OA}^2-\text{AD}^2;$$
$$\therefore\ (15-x)^2=15^2-9^2=225-81=144.$$

Taking the square root of each side,

$$15-x=\pm 12,$$
$$\therefore\ x=15\mp 12=3\text{ or }27.$$

26. $$\frac{(2x)^2\times{}\cdot 7854}{4}-\frac{x^2}{2}=8,$$
$$\therefore\ x^2\times{}\cdot 7854\times 2-x^2=16,$$

Completing the square by adding 8^2 or 64 to each side,

$$x^2+16x+8^2=64+64=128,$$

$$\therefore\ x+8=\sqrt{128}=11\cdot3137,$$

$$\therefore\ x=11\cdot3137-8=3\cdot3137.$$

32. Let x=one of the equal sides ; then
$20-2x$=the hypotenuse ;

$$\therefore\ (20-2x)^2=x^2+x^2$$

$$\therefore\ 400-80x+4x^2=2x^2,$$

$$\therefore\ 2x^2-80x=-400,$$

$$\therefore\ x^2-40x=-200.$$

Completing the square by adding 20^2 or 400 to each side,

$$x^2-40x+20^2=-200+400=200$$

$$\therefore\ x-20=\pm\sqrt{200}=\pm14\cdot142.$$

Taking the + sign of the quantity, we have

$$x=14\cdot142+20=34\cdot142.$$

Taking the − sign of the quantity, we have

$$x=-14\cdot142+20=5\cdot858.$$

Therefore 5·858 is the side required, the other root not admitting of a consistent interpretation. On questions of this kind, see Art. **95** of the Author's 'Principles of Geometry and Mensuration, &c.'

33. Let x=the side; then
$x-4$=the perpendicular,
and $\frac{x}{2}$= half the base ;

$$\therefore\ \left(\frac{x}{2}\right)^2+(x-4)^2=x^2,$$

$$\therefore\ \frac{x^2}{4}+x^2-8x+16=x^2,$$

$$\therefore\ x^2-32x=-64.$$

Completing the square by adding 16^2 or 256 to each side,

$$x^2-32x+16^2=-64+256=192,$$

$$\therefore\ x-16=\pm\sqrt{192}\ =\pm13\cdot8564.$$

Taking the + sign of this result, we have

$$x=+13{\cdot}856+16=29{\cdot}856;$$

taking the − sign of the result, we have

$$x=-13{\cdot}8564+16=2{\cdot}143.$$

35. Let x=one of the sides,
and y=the common ratio, then
xy=the other side, and xy^2=the hypotenuse.

From the property of the right-angled triangle, we have

$$(xy^2)^2=x^2+(xy)^2,$$
$$\therefore\quad x^2y^4=x^2+x^2y^2,$$
$$\therefore\quad y^4=1\ +y^2,$$
$$\therefore\ y^4-y^2=1.$$

Completing the square by adding $(\frac{1}{2})^2$ or $\frac{1}{4}$ to each side,

$$y^4-y^2+\tfrac{1}{4}=1+\tfrac{1}{4}=\tfrac{5}{4},$$

taking the square root, $y^2-\frac{1}{2}=\sqrt{\frac{5}{4}}=\frac{\sqrt{5}}{2}$,

$$\therefore\ y^2=\frac{\sqrt{5}}{2}+\tfrac{1}{2}=\tfrac{1}{2}(\sqrt{5}+1)=\tfrac{1}{2}(2{\cdot}236+1)=1{\cdot}618,$$
$$\therefore\ y=\sqrt{1{\cdot}618}=1{\cdot}272.$$

Again, the area of the triangle $=\frac{1}{2}x\times xy$; but, by the question, this is 36,

$$\therefore\ \tfrac{1}{2}x\times xy=36.$$

Substituting 1·272 for y, we get

$$\tfrac{1}{2}x^2\times 1{\cdot}272=36,$$
$$\therefore\ x^2=\frac{36\times 2}{1{\cdot}272}=56{\cdot}603,$$
$$\therefore\ x=\sqrt{56{\cdot}603}=7{\cdot}52,\text{ one side;}$$

and the other side$=xy=7{\cdot}52\times 1{\cdot}272=9{\cdot}56+$.

37. Let x=AP; then $100-x$=PB.

From the right-angled triangles APD and PBC, we have
$DP^2=AP^2+AD^2=x^2+30^2=x^2+900$, and
$PC^2=PB^2+BC^2=(100-x)^2+70^2$
$=10000-200x+x^2+4900=x^2-200x+14900;$

but, by the question, DP=PC,

$$\therefore\ x^2+900=x^2-200x+14900,$$
$$\therefore\ 200x=14900-900=14000,$$
$$\therefore\ x=\frac{14000}{200}=70.$$

38. From the foregoing solution, we find

$$PC^2-DP^2=x^2-200x+14900-(x^2+900)$$
$$=-200x+14000;$$

but, by the question, this is equal to 400,

$$\therefore\ -200x+14000=400,$$
$$\therefore\ 200x=13600,$$
$$x=\frac{13600}{200}=68 \text{ ft.}$$

39. Let x=the height of the tower, in feet; then by similar triangles, we have

$$4:6::x:90,$$
$$\therefore\ 6x=90\times4=360,\ \therefore\ x=\tfrac{360}{6}=60 \text{ ft.}$$

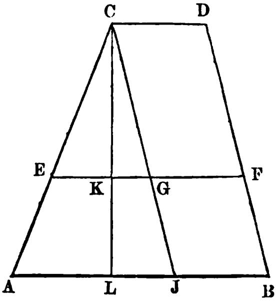
40. Let FGE be the equilateral triangle, and ADCB the inscribed square; then DCE will be an equilateral triangle, and therefore DE=DC.

Let x=AD=DC=DE; then FD=FE−DE=$8-x$; and the perpendicular height, EP, of the triangle FGE= $\sqrt{8^2-(\frac{8}{2})^2}=\sqrt{48}$. Now by the similar triangles FDA and FEP, we have

$$FD:DA::FE:EP, \text{ that is,}$$
$$8-x:x::8:\sqrt{48}$$
$$\therefore\ 8x=(8-x)\sqrt{48},$$
$$\therefore\ 8x=8\sqrt{48}-\sqrt{48}\times x,$$
$$\therefore\ 8x+\sqrt{48}\times x=8\sqrt{48},$$
$$\therefore\ x=\frac{8\sqrt{48}}{8+\sqrt{48}}=\frac{8\times6{\cdot}9282}{8+6{\cdot}9282}$$
$$=\frac{55{\cdot}4256}{14{\cdot}9281}=3{\cdot}713 \text{ nearly}$$

41. Let ABDC represent the deal in the form of a trapezoid, EFDC the part cut off, CJ parallel to DB, and CKL perpendicular to AB or to EF; then CD=GF=JB=1 ft., AB=$1\frac{1}{2}$ ft.=$\frac{3}{2}$ft., CL=8 ft., AJ=AB−JB=$\frac{3}{2}-1=\frac{1}{2}$ ft.

Let x=CK, the length of the part cut off, in feet; then by the similar triangles ACJ and ECG, we have



$$\text{CL} : \text{AJ} :: \text{CK} : \text{EG, that is,}$$

$$8 : \tfrac{1}{2} :: x : \text{EG}=\frac{\frac{1}{2}\times x}{8}=\frac{x}{16},$$

$$\therefore \text{EF}=\text{GF}+\text{EG}=1+\frac{x}{16}.$$

Area of the trapezoid EFDC in sq. ft.$=\frac{1}{2}\left(1+\frac{x}{16}+1\right)x$

$$=\tfrac{1}{2}\left(2+\frac{x}{16}\right)x=\tfrac{1}{32}\left(x^2+32x\right).$$

But, by the question, this area is equal to 2 sq. ft.,

$$\therefore \tfrac{1}{32}\left(x^2+32x\right)=2,$$

$$\therefore x^2+32x=64.$$

Completing the square by adding 16^2 or 256 to each side,

$$x^2+32x+16^2=64+256=320,$$

$$\therefore x+16=\sqrt{320}=17{\cdot}8885,$$

$$\therefore x=1{\cdot}8885 \text{ ft.}$$

42.
$$xy=1200 \quad .\;.\;.\;.\; (1.)$$
$$x^2+y^2=2500 \quad .\;.\;.\;.\; (2.)$$

Adding twice eq. (1.) to eq. (2.), and taking the square root of the result, we get

$$x+y=\sqrt{2500+2400}=\sqrt{4900}=70 \quad .\;.\;.\;.\; (3.)$$

Subtracting twice eq. (1.) from eq. (2.), and taking the square root of the result, we get

$$x-y=\sqrt{2500-2400}=\sqrt{100}=10 \quad . \ . \ . \ . \ (4.)$$

Adding (4.) and (3.), $2x=80$, $\therefore\ x=40$;
Subtr. (4.) from (3.), $2y=60$, $\therefore\ y=30$.

43. Let x=the perpendicular, CP. (See *fig.* to Prob. 6.)

$$AB=\sqrt{9^2+12^2}=\sqrt{81+144}=\sqrt{225}=15.$$

Double the area of the triangle$=15\times x$; but we have also
„ „ „ $=9\times12=108$,

$$\therefore\ 15\times x=108,\ \therefore\ x=\tfrac{108}{15}=7{\cdot}2.$$

44. Putting x and y for the two sides respectively; then

$$x^2+\left(\frac{y}{2}\right)^2=4^2=16$$

$$y^2+\left(\frac{x}{2}\right)^2=6^2=36.$$

By reduction these equations become,

$$4x^2+y^2=64 \quad . \ . \ . \ . \ (1.)$$
$$4y^2+x^2=144 \ . \ . \ . \ . \ (2.)$$

Subtracting eq. (2.) from four times eq. (1.), we get

$$16x^2-x^2=64\times4-144=122,$$
$$\therefore\ 15x^2=122,$$
$$\therefore\ x=\sqrt{\frac{122}{15}}=\sqrt{7{\cdot}466}=2{\cdot}732.$$

Subtracting eq. (1.) from four times eq. (2.), we get

$$15y^2=576-64=512,$$
$$\therefore\ y=\sqrt{\frac{576}{15}}=\sqrt{34{\cdot}133}=5{\cdot}842.$$

45. Let x=the radius of the inscribed circle; then

$$\text{The hypotenuse}=\sqrt{12^2+16^2}=\sqrt{400}=20.$$

Double the area of the triangle$=12\times16$; but we have also
„ „ „ $=12\times x+16\times x+20\times x=48x$

$$\therefore\ 48x=12\times16,\ \therefore\ x=\frac{12\times16}{48}=4.$$

47. Let x=the breadth in yards, then

$$180-2x=\text{the side of the interior square;}$$

$\therefore$ area of the coach road$=180^2-(180-2x)^2=720x-4x^2$.

$$\therefore 720x-4x^2=1424.$$

Changing all the signs and dividing by 4, we get

$$x^2-180x=-356.$$

Completing the square by adding 90^2 or 8100 to each side,

$$x^2-180x+90^2=-356+8100=7744,$$

$$\therefore x-90=\sqrt{7744}=\pm 88,$$

$$\therefore x=\pm 88+90$$

$$=178 \text{ or } 2.$$

Hence the breadth required is two yards, the other value of x being inconsistent with the conditions of the particular problem.

48. Let x=the diameter, in inches.

$$\text{Volume of the ball in c. ft.}=\frac{x^3\times\cdot 5236}{1728};$$

$$\therefore \text{ weight of the ball in oz.}=\frac{x^3\times\cdot 5236\times\overset{100}{\cancel{7200}}}{\underset{24}{\cancel{1728}}}=\frac{x^3\times 13\cdot 09}{6};$$

but the weight of the ball$=24\times 16$ oz.$=384$ oz.;

$$\therefore x^3\times\frac{13\cdot 09}{6}=384$$

$$\therefore x^3=\frac{384\times 6}{13\cdot 09}=176\cdot 01,$$

$$\therefore x=\sqrt[3]{176\cdot 01}=5\cdot 60+, \text{ inches.}$$

49. Let x=the diameter of the sphere, in inches; then

$$\text{Volume of the sphere in c. ft. }=\frac{x^3\times\cdot 5236}{1728};$$

$\therefore$ Weight of the displaced water, in oz.

$$=\frac{x^3\times\cdot 5236\times 1000}{1728}=\frac{x^3\times 523\cdot 6}{1728}.$$

Weight of the copper, in oz. $=\frac{8915}{1728}$;
but the weight of the displaced water is equal to the weight of the body,

$$\therefore \frac{x^3\times 523{\cdot}6}{1728}=\frac{8915}{1728},$$

$$\therefore x^3=\frac{8915}{523{\cdot}6}=17{\cdot}02,$$

$$\therefore x=\sqrt[3]{17{\cdot}02}=2{\cdot}57+,\text{ inches.}$$

50. Let $x=$the no. cubic inches of cork ; then

$$\text{Weight of the cork, in oz.}=\frac{x\times 240}{1728};$$

$$\therefore \text{Weight of the cork and copper, in oz.}=\frac{x\times 240}{1728}+\frac{8915}{1728}$$

Volume of the displaced water, in c. in=

$$\text{Vol. copper}+\text{Vol. cork}=1+x,$$

$$\therefore \text{Weight of the displaced water, in oz.}=\frac{(1+x)\ 1000}{1728},$$

$$\therefore \frac{(1+x)\ 1000}{1728}=\frac{x\times 240}{1728}+\frac{8915}{1728},$$

$$\therefore 1000+1000x=240x+8915,$$

$$\therefore 760x=7915,$$

$$\therefore x=\frac{7915}{760}=10{\cdot}41\text{ c. in.}$$

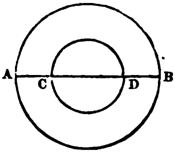
51. Let $x=$the side $AC=AB=BC$, (see *fig.* to Prob. 18.)
$3=BD$, the perpendicular height,
and $\frac{x}{2}=AD=DC$; then .
$AB^2=AD^2+BD^2$, that is,

$$x^2=\left(\frac{x}{2}\right)^2+3^2,$$

$$\therefore 4x^2=x^2+36,$$

$$\therefore 3x^2=36,\ \therefore x^2=12,\ \therefore x=\sqrt{12}.$$

52. Let $x=AB$, the diameter of the outer circle in ft.; then



$x-6=CD$, the diameter of the inner circle in ft.; then

Area of the walk, in sq. ft. $=x^2\times\cdot7854-(x-6)^2\times\cdot7854$;

but the area of the walk is 150 sq. ft.,

$$\therefore\ x^2\times\cdot7854-(x-6)^2\times\cdot7854=150,$$

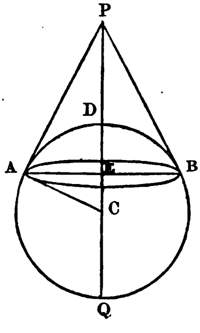
$$\therefore\ x^2-(x-6)^2=\frac{150}{\cdot7854}$$

$$\therefore\ x^2-x^2+12x-36=\frac{150}{\cdot7854},$$

$$\therefore\ 12x=\frac{150}{\cdot7854}+36,$$

$$\therefore\ x=\frac{150}{12\times\cdot7854}+3=18\cdot91\text{ ft.}$$

53. Let C represent the earth's centre, DP the height above its surface, and ABD the surface seen; then PA and PB will be tangents to the circle ADBQ, and the line DP produced will pass through the centre C.



Let $x=ED$, the height of the segment, ABD, of the sphere; $d=$the diameter DQ; then by Prob. 24, page 175, of the Author's 'Geometry, Mensuration, &c.,' we have

Surface of the sphere $=d^2\times3\cdot1416$;

Surface of the spherical segment $ABD=d\times3\cdot1416\times x$;

but, by the question, the latter is $\frac{1}{3}$ the former

$$\therefore\ d\times3\cdot1416\times x=\tfrac{1}{3}d^2\times3\cdot1416,$$

$$\therefore\ x=\tfrac{1}{3}d=ED;$$

$$\therefore\ \text{CE}=\text{CD}-\text{ED}=\frac{d}{2}-\frac{d}{3}=\frac{d}{6};$$

$$\text{AE}^2=\text{AC}^2-\text{CE}^2=\left(\frac{d}{2}\right)^2-\left(\frac{d}{6}\right)^2=\frac{d^2}{4}-\frac{d^2}{36}=\frac{2}{9}d^2.$$

Now by the similar triangles ACE and AEP, we have

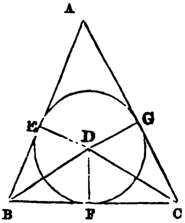
$$\text{CE} : \text{AE} :: \text{AE} : \text{EP},$$

$$\therefore\ \text{EP}=\frac{\text{AE}^2}{\text{CE}}=\frac{\frac{2}{9}d^2}{\frac{d}{6}}=\frac{12d^2}{9d}=\frac{4}{3}d,$$

$$\therefore\ \text{DP}=\text{EP}-\text{DE}=\tfrac{4}{3}d-\tfrac{1}{3}d=d=\text{DQ}.$$

That is, the height above the earth's surface will be equal to its diameter.

54. Let ABC be the triangle, D the centre of the inscribed circle EFG, and DF=DE=DG, the radii, &c.



Let $x=\text{AB}=\text{BC}=\text{AC}$; then

$$\text{AF}=\sqrt{\text{AB}^2-\text{BF}^2}=\sqrt{x^2-\frac{x^2}{4}}=\frac{x}{2}\sqrt{3}.$$

The area triangle $\text{ABC}=\frac{1}{2}\text{BC}\times\text{AF}$

$$=\frac{x}{2}\times\frac{x}{2}\sqrt{3}=\frac{x^2}{4}\sqrt{3};$$

but we have also,

The area triangle ABC = 3 times area triangle BDC

$$=3\times\tfrac{1}{2}\text{BC}\times\text{FD}=3\times\frac{x}{2}\times10=15x;$$

$$\therefore\ \frac{x^2}{4}\sqrt{3}=15x$$

$$\therefore\ x=\frac{15\times4}{\sqrt{3}}=\frac{3\times5\times4}{\sqrt{3}}=\sqrt{3}\times5\times4$$

$$=20\times\sqrt{3}=20\times1{\cdot}732=34{\cdot}64.$$

55. From eq. (2.), $w=37-z=37-12=25$;
Substituting 12 for z, and 25 for w in (4.) and (3.), we get

$$x^2+y^2=25^2=625\ \ldots\ldots\ (3.)$$

$$xy=25\times12=300\ \ldots\ldots\ (4.)$$

Subtracting twice (4.) from (3.), and taking the sq. root of the result,

$$x-y=\sqrt{625-600}=\sqrt{25}=5 \;.\;.\;.\;.\; (5.)$$

Adding this equation to eq. (1.), $2x=40$, $\therefore\ x=20$
Subt. (5.) from (1.), $2y=30$, $\therefore\ y=30$.

56. Let $x=\text{BC}$, $y=\text{AC}$, $w=\text{AB}$, and $z=\text{CP}$ (see *fig.* to Prob. 6.); then, from the question, we have

$$x-y=10 \;.\;.\;.\;.\; (1.)\,;\quad w-z=26 \;.\;.\;.\;.\; (2.)$$

From the right-angled triangle ABC, we have

$$x^2+y^2=w^2 \;.\;.\;.\;.\;.\; (3.)$$

Double the area of the triangle $\text{ABC}=\text{BC}\times\text{CA}=xy$;
also „ „ „ „ $=\text{AB}\times\text{PC}=wz$;

$$\therefore\ xy=wz \;.\;.\;.\;.\;.\; (4.)$$

Subtracting the square of (1.) from (3.), we get

$$2xy=w^2-100.$$

Substituting the value of xy given in (4.), we get

$$2wz=w^2-100,$$
$$\therefore\ w^2-2wz=100.$$

Subtracting this equation from the square of (2.),

$$z^2=26^2-100=676-100=576,$$
$$\therefore\ z=\sqrt{576}=24.$$

Substituting in (2.), $w-24=26$, $\therefore\ w=50$.

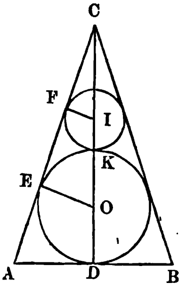
Substituting these values of z and w in (4.) and (3.),

$$xy=50\times24=1200 \;.\;.\; (5.),$$
$$x^2+y^2=50^2=2500 \;.\;.\;.\; (6.).$$

Adding twice (5.) to (6.), and taking the square root of the result,

$$x+y=\sqrt{2500+2400}=70 \;.\;.\; (7.).$$

Adding (1.) and (7.) $2x=80$, $\therefore\ x=40$;
subtracting (1.) from (7.), $2y=60$, $\therefore\ y=30$.



57. Let ABC represent the isosceles triangle, CD the perpendicular, OE and IF the radii of the inscribed circles, &c.

Let $x = \text{IC}$; then

$\text{OC} = \text{IC} + \text{IK} + \text{OK} = x + 4 + 8 = x + 12$.

By the similar triangles OEC and IFC, we have

$$\text{OC} : \text{OE} :: \text{IC} : \text{IF}, \text{ that is,}$$

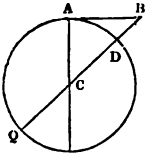
$$x + 12 : 8 :: x : 4,$$

$$\therefore 8x = 4(x + 12) = 4x + 48,$$

$$\therefore 4x = 48, \therefore x = 12 = \text{IC};$$

$$\therefore \text{CD} = \text{IC} + \text{IK} + \text{KD}$$

$= 12 + 4 + 2 \times 8 = 32$, the perpendicular required.



58. Let A represent the place of observation, BD the elevation of the mountain, C the centre of the earth; then AB will be a tangent to the earth's surface at A.

Let $x = \text{DQ}$, the diameter of the earth; then from Cor. 2., page 58, Tate's 'Geometry and Mensuration,' we have

$$\text{BQ} \times \text{BD} = \text{AB}^2;$$

but $\text{BD} = h$, $\text{AB} = d$, and $\text{BQ} = \text{DQ} + \text{BD} = x + h$,

$$\therefore (x + h)h = d^2,$$

$$\therefore xh + h^2 = d^2,$$

$$\therefore \quad x = \frac{d^2 - h^2}{h}, \text{ the diameter required.}$$

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